Contents

1 Overview of main research directions

1. **Inverse problems, scattering theory and cloaking**

An important research field for me is on scattering theory with application to cloaking, see Sections 2.1.2, and 2.1.1. Among many open questions, the mathematical analysis of the transformation based approximate cloaking scheme may lead one to better understand and address the challenges appearing in the inverse scattering theory for highly lossy media. Also, the analysis of non-metamaterial based cloaking schemes is very important as this may lead to cost efficient acoustic or seismic cloaks.

Another interesting research project, which I worked on and I plan to reconsider in the future is on inverse problems for electromagnetism, i.e identification of heterogeneities for electromagnetism in lossy inhomogeneous media, for the non-monochromatic case based only on boundary measurements, see Section 2.4.

2. **Variational analysis and optimization with application to the study of the multi-scale phenomena in material science**

The study of new emerging materials, such as, meta-materials or dynamic materials, fascinates me from mathematical as well as physical point of view and during my postdoctoral years I worked on various projects related to the study of such materials and their applications, (e.g., cloaking theory for Helmholtz, Maxwell and Elasticity equations, negative index of refraction and perfect absorbers, wave propagation and energy concentration in dynamic materials, macroscopic characterization of meta-materials and dynamic materials), see Sections 2.1.1, 2.2, and 2.3.

Optimization theory for cloaking recently produced spectacular numerical results [41] (see also [42]), and I would like to gain a better mathematical understanding of this because it may lead one to understand the limitation of cloaking when, for example, one is bound to use certain type of laminated materials for the cloak. Also, a good analytical understanding of the cloaking scheme through optimization may shine new light on other deep open questions in cloaking theory.

3. **Homogenization theory for partial differential equations with applications**

My Ph.D. thesis was focused on various problems in the multi-scale analysis of elliptic PDE's with either non-smooth coefficients or non-smooth domains. A few projects I worked on during my Ph.D studies are, error estimates for homogenization of linear elliptic PDE's in general convex domains and with non-smooth coefficients, asymptotic analysis of linear elliptic PDE's in perforated domains, multi-scale phenomena in earthquake modelling), see Sections 3.1, 3.2 and 3.3. I am interested in error estimates results for the homogenization of nonsmooth elliptic problems and in Γ-convergence and its applications to the limit analysis of discrete models, see Section 2.2.

2 Current and future research

2.1 Cloaking theory

2.1.1 Approximate cloaking of acoustic and electromagnetic waves

In the academical year 2007-2008, during my postdoctoral year at Rutgers, I started working on cloaking theory. This subject is of great interest for its important applications, and has recently been addressed by many authors, in the physical and mathematical communities, (see the reviews [17], [18] and references therein). There are several approaches towards cloaking in the literature, among which one can list,:

- anomalous localized resonance introduced in [11]
- optical conformal mapping introduced in [9]
- change of variables scheme, introduced in ([7], [8], [10])
- active cloaking (interior cloak [12], exterior cloak [15], [16])
- complimentary media cloaking ([14]

One of my research directions in this area is focused on the cloaking of acoustic and electromagnetic waves in the setting of the change of variables scheme.

In the general framework of inverse problems, a definition of cloaking is as follows:

Definition 2.1. Suppose we have an initial configuration formed by material Y (to be cloaked), which (without loss of generality) occupies the unit ball centered in origin, B_1, and material X (of the cloak), located in the region $\Omega \setminus B_1$. We say that material X cloaks material Y, if material X does not depend on material Y and if the Neumann to Dirichlet map associated to the PDE (Helmholtz or Maxwell) in the initial configuration is identical to the Neumann to Dirichlet map associated to the same PDE in a homogeneous medium.

Following the pioneer idea of Dolin presented in [7], which observed that suitable transformation of variables in Maxwell equations renders equivalent media, i.e., the Neumann to Dirichlet map corresponding to the two media are the same, the change of variable scheme for cloaking was first introduced by Greenleaf, Lassas and Uhlmann in [8], for zero frequency, and by Pendry, Schurig and Smith in [10], for finite frequency, (see also [19]).

Let B_R denote the ball of radius R centered in the origin and consider $\Omega = B_2$. For example, for the Helmholtz equation at fixed frequency ω, the general idea behind **change of variables scheme** is that using a change of variables given by $y = F(x)$, where $F : B_2 \to B_2$ is invertible and $F(x) = x$ on ∂B_2, the equation (2.1)

$$\nabla \cdot (A \nabla u) + \omega^2 q u = 0 \text{ in } B_2 \tag{2.1}$$

where (A, q) are material parameters, is transformed into an equivalent problem,

$$\nabla \cdot (F_* A \cdot \nabla u) + \omega^2 F_* q \cdot u = 0 \text{ in } B_2 \tag{2.2}$$

where the transformed material parameters (F_*A, F_*q) are naturally obtain from the change of variable process and are explicit functions of the initial material parameters (A, q) and the jacobian matrix of the transformation F. Moreover, we have that the Dirichlet to Neumann maps, $\Lambda_{A,q}$, Λ_{F_*A,F_*q}, or in other words, the boundary measurrements maps, (voltage to current maps for the conductivity equation) associated to the two problems (2.1) and (2.2), respectively, are identical.

In [19], [10], the authors applied the above idea by using the following radially symmetric transformation of variables, $F:\ B_2 \to B_2$

$$F(x) = \left(1 + \frac{1}{2}|x|\right)\frac{x}{|x|} \tag{2.3}$$

The map F blows up the origin to the ball B_1, and is equal to the identity on the boundary of the domain B_2. By using this change of variables, the Neumann to Dirichlet map will not change, and moreover Dolin's idea implies that the effect of the material located in B_1 on the boundary measurements will be the same as the effect of a point (in this case the origin) on the same boundary measurements. Thus cloaking can be achieved.

The only big problem is that the transformation F defined at (2.3), has a point singularity in the origin and its inverse has singularities around the boundary of B_1. The analysis is possible but with a careful definition of the notion of weak solution for such singular problems (see [19]). This scheme was extensively studied in the physical and mathematical literature (see [18] and references therein), and it is well known now that an ideal perfect cloak resulting from this scheme will be unstable (see [13]), and it requires exotic materials (e.g. with zero or infinite eigenvalues of conductivity, permittivity or permeability tensor) not available in general.

In a joint effort to develop a cloaking scheme which do not make use of materials with such extreme properties, together with R. V. Kohn, M. Vogelius, and M. Weinstein (see [20]), building up on an early work [21], we analyzed the behavior of the cloak produced by a regularized version of the previous singular change of variable scheme, for Helmholtz equation at fixed frequency. Our model correspond in to TM guided waves in 2D and to acoustic waves in 3D.

The novelty of our scheme is that although it uses the same idea as before it is based on a more regular transformation of variables, given by a map F_ρ which blows up B_ρ, to the cloaked region, in this case $B_{\frac{1}{2}}$, maps the annuli $B_{2\rho} \setminus B_\rho$ into the annuli $B_1 \setminus B_{\frac{1}{2}}$ and is equal to the identity on the boundary of the domain.

One of the main ideas behind our results, is that in order to deal with resonances, we considered a lossy material with loss parameter β (with the effect of damping) distributed in the annuli $B_1 \setminus B_{\frac{1}{2}}$ as part of our anisotropic cloak.

Using the map F_ρ in the change of variables scheme described above, the problem of "approximately" cloaking an arbitrary material located in B_1 becomes equivalent, with the problem of minimizing the effect on the boundary data (that is, on the Dirichlet to Neumann map), of a small inhomogeneity of size ρ surrounded by a lossy layer of the same size, embedded in homogeneous media. Mathematically this translates into the fact that for high accuracy of our approximate cloaking we will need to have a small operatorial norm of the difference between the Dirichlet to Neumann map associated to homogeneous media and the Dirichlet to Neumann map associated to the homogeneous media surrounding a small inhomogeneity of size ρ coated with a lossy layer of the same size.

We analytically proved that the optimal loss parameter in the layer should be of order $O(\rho^{-2})$ and that in this case the cloak works with an accuracy of $O(\frac{1}{|\log \rho|})$ in 2D and with accuracy of

$O(\rho)$ in 3D.

We also produced strong 2D and 3D numerical evidence to support the optimality of these results. For the numerical computations we assumed homogeneous and isotropic media, an incoming plane wave and pairs of **resonant coefficients** (A_ρ, q_ρ) corresponding to resonant materials, (or singular materials).

In the following plots $E_\rho(\beta)$ denotes the normalized $H^{-\frac{1}{2}}$ norm of the difference between the two Dirichlet to Neuamnn maps, one corresponding to the homogeneous media another corresponding to homogeneous media surrounding a small inhomogeneity of size ρ coated with the lossy layer with a loss parameter of order $O(\rho^{-2})$, where the normalization is done with respect to the incoming plane wave and the 2D (3D) respective error, i.e., $O(\frac{1}{|\log \rho|})$ $(O(\rho))$.

The color code in the plots, is red for the 2D case and blue for the 3D case.

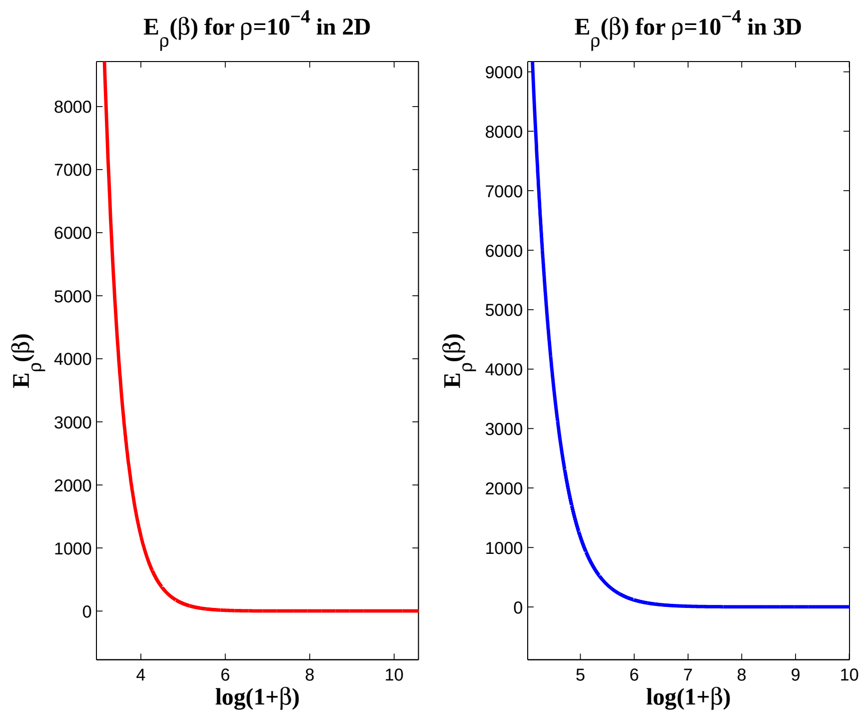

Figure 1: The influence of the lossy layer on the estimate

The x-axis in and Figure 1 and Figure 2 supports values of $log_{10}(1+\beta)$ for $0 < \beta < 2\rho^{-3}$.

Figure 1, shows the plot of $E_\rho(\beta)$ as a function of β, for fixed $\rho = 10^{-4}$ and for a pair of **"ressonant coefficients"**. One can see clearly the importance on the lossy layer in our approximate cloak behavior, by observing that with no layer considered, i.e. with $\beta \approx 0$, $E_\rho(\beta)$ blows up.

In Figure 2, assuming the same conditions as for Figure 1, we show the plot of $E_\rho(\beta)$ as a function of β, zoomed in around $\beta = \rho^{-2}$. It is immediate now that $E_\rho(\beta)$ attains its minimum around $\beta \approx \rho^{-2}$ and, because we considered **"resonant coefficients"**, this is a good numerical indication that $\beta \approx \rho^{-2}$ is optimal in our analysis.

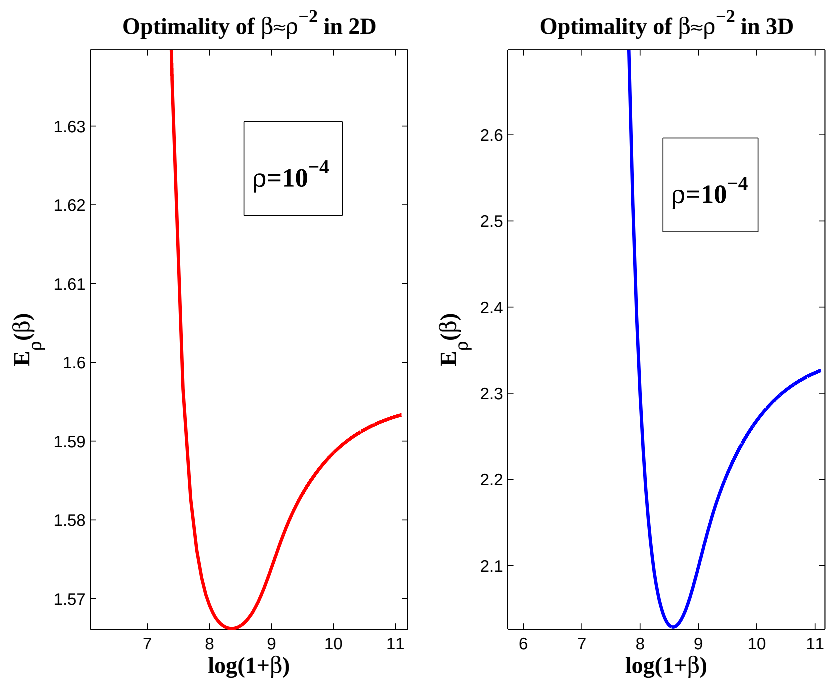

Figure 2: Optimality of $\beta \approx \rho^{-2}$ in our estimate

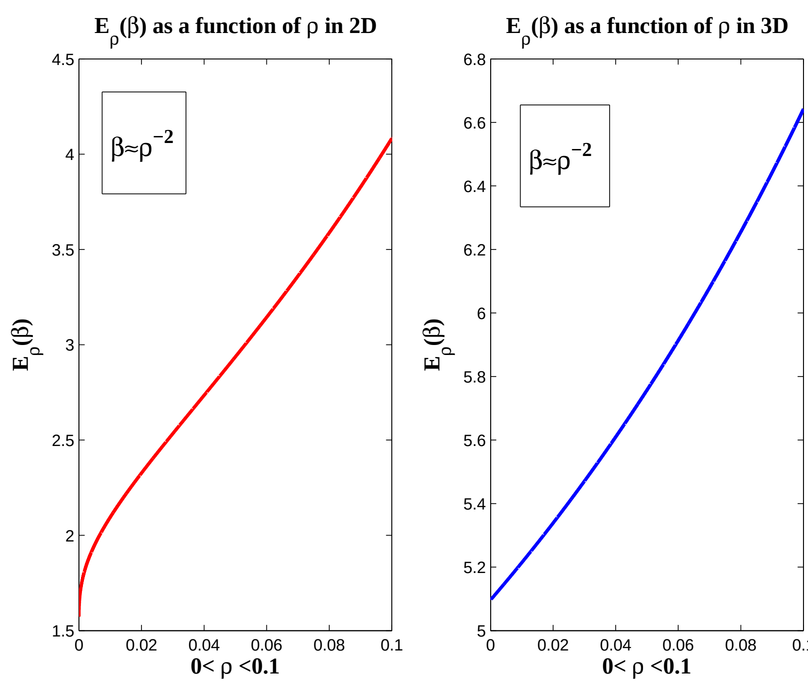

Figure 3: The evolution of $E_\rho(\beta)$ with respect to $\rho << 1$ for $\beta \approx \rho^{-2}$

Figure 3 shows the plot of $E_\rho(\beta)$ as a function of ρ, for beta $\beta \approx \rho^{-2}$ and for a pair of **"resonant coefficients"** One can see that when $\rho << 1$, $E_\rho(\beta)$ is close to a constant in 2D as well as in 3D case, which agrees with our theoretical estimate.

A few important **future projects** are:

i) To study the dependence of our estimates on frequency.
ii) To extend our results for the general case of dissipative cloaks.
iii) The study of the stability of our cloak and practical. feasibility.
iv) Does the near cloak work for active materials inside the cloak region?
v) The generalization of our results to the full Maxwell equations
vi) The study of the cloaking scheme for the TE guided waves.

2.1.2 Active exterior cloaking

During the current academical year, in a joint effort with professor G.W. Milton, and F.Guevara-Vasquez, I started working on a new cloaking scheme, which we called "active exterior cloak".

The main reasons behind our effort were to propose a cloaking scheme, which would not require the use of meta-materials, thus being easier to implement then the previous schemes, which would work over a broadband, thus cloaking objects from an incoming pulse, and which would offer the advantage that the cloaked area will have information about the exterior while being "invisible".

In the quasistatic case, our problem can be mathematically formulated as follows,

Let $0 < \delta << 1$, and a, c, R such that, $\delta < c$ and $0 < c + a < R$. Find $g \in H^2(B_R(0) \setminus B_\delta(0))$ such that the following problem is well-posed,

$$\begin{cases} \Delta u = 0 & \text{in } B_R(0) \setminus B_\delta(0) \\ u = g + f & \text{on } \partial B_\delta(0) \\ u \approx f & \text{for } |x| \geq R \\ u \approx 0 & \text{in } B_a(c) \end{cases} \tag{2.4}$$

where f is a harmonic potential in the whole space, and $B_x(y)$ denotes the ball of radius x centered in y.

In the 2D case, with the assumption that the cloaked region is small enough, we were able to rigourously prove in [15], that cloaking is possible, i.e., there exists a function g such that the problem (2.4) is well-posed. This is equivalent to saying that, with the above assumptions, there exists a field g to be generated by the device located on $\partial B_\delta(0)$, such that the potential will be almost zero in the "cloaked" region $B_a(c)$ and the total scattering effect on ∂B_R will be very small. One can see this from the Fig 4, where the behavior of our cloaking scheme is shown. The black disk in the plot represents the cloaked region and the exterior dashed circle represents the boundary of exterior measurements, $\partial B_R(0)$. The plot shows that, when the device located around the origin is active, an incoming plane wave (in this figure we assume a quasistatic regime, i.e., waves of small frequency) propagates through the media almost as in an empty space.

Our method was tailored for dimension two and we could not generalized it to higher dimensions. In an attempt to generalize the result to the 3D quasistatics one can formulate the problem equivalently as follows,

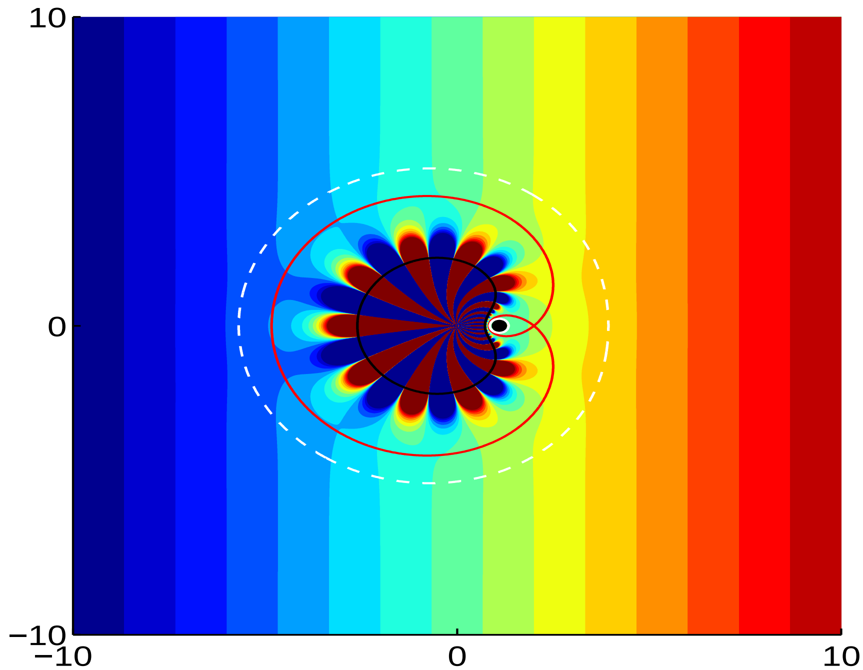

Figure 4: Active exterior cloaking for quasi-statics

Let $\epsilon << 1$. Let $\mathcal{C}$ be the class of functions $h \in H^2(B_R(0) \setminus B_\delta(0))$ satisfying the following conditions,

$$\begin{cases} \Delta h = 0 & \text{in } B_R(0) \setminus B_\delta(0) \\ ||h||_{L^2(\partial B_R(0))} = O(\epsilon) \\ ||\frac{\partial h}{\partial r}||_{L^2(\partial B_R(0))} = O(\epsilon) \end{cases} \tag{2.5}$$

where $O(\epsilon)$ represents the class of functions of sublinear growth with respect to ϵ. Assuming that a, c, R and f are given as in (2.4), we have that a field g needed at the device to guarantee approximate cloaking within an error of $O(\epsilon)$ exists if one proves that

$$\inf_{h \in \mathcal{C}} ||h + f||_{L^2(B_a(c))} = O(\epsilon)$$

Moreover, if one wants to build a causal cloak one would need to prove that the scheme is independent of the incoming "wave" f, and in this regard, instead of solving an infimum problem as above one needs to show that,

$$\sup_f \inf_{h \in \mathcal{C}} ||h + f||_{L^2(B_a(c))} = O(\epsilon)$$

The analysis of this approach for the 2D case and its adaptation to the 3D case is part of my immediate **future research plans**.

In [16], we also analyzed the feasibility of our cloaking scheme for the Helmholtz equation (narrow band and broadband), which models the propagation of acoustic waves and guided electromagnetic waves. Our results show that active exterior cloaking is possible in this case, if one uses at least three active devices positioned around the region to be cloaked, see Fig 5. The plot in Fig 5 shows that if the devices are active, an incoming plane wave propagates through the region containing a kite-shaped object (to be cloaked) and the three active devices almost as through the empty space.

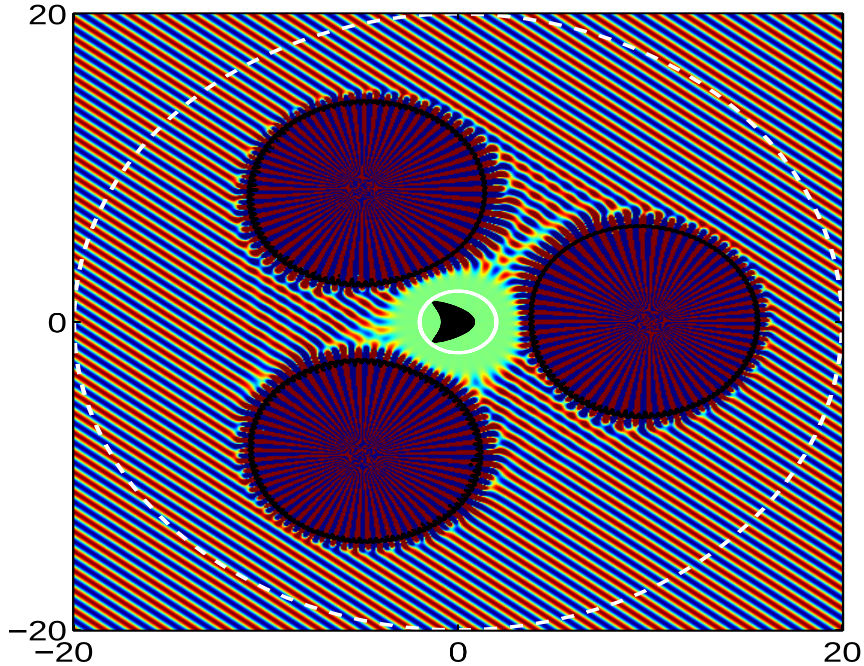

Figure 5: Active exterior cloaking for Helmholtz equation

So far, for the Helmholtz equation, we produced strong numerical evidence to support our results, [15], [16]. Currently, we are working on an integral equations approach to fully mathematically understand the scheme.

Thus, the mathematical analysis of the active exterior cloak for Helmholtz equation is one of the **current research projects** I am involved in.

There are many interesting open questions regarding the active cloak scheme, such as, the practical feasibility of the model, the causality of the scheme, the necessity of three devices for the finite frequency case, the stability of the scheme and the possibility to cloak subregions of non-simple connected regions, to name a few. These questions are part of my **future research plans**.

2.2 Elastodynamic Meta-materials

Another important focus of my current research is on meta-materials. Meta-materials are generally understood as man-made high contrast composites, which on the macroscopic scale may manifest exotic properties not found in nature. It is known that meta-materials can macroscopically manifest extreme properties, such as, a negative Poison ratio, having high shear modulus and low bulk modulus ([35], [36]), negative refractive index ([37]), negative effective density ([38]). Moreover, one can construct anisotropic composites having any desired linear positive definite elasticity tensor ([39] and references therein). Thus, a very interesting feature of meta-materials is that their macroscopic behavior may be completely unnatural, and it may be governed by equations which would be totaly different then those valid at the micro-scale. I am extremely interested in studying the continuum description of meta-materials which is a very important step in the understanding of their design and of various exotic properties they manifest. In this regard, one of my recent projects, [40], was focused on the complete description (2D and 3D as well) of the frequency dependence of all possible response matrices for a discrete elastodynamic network under the external action of a balance set of forces and on the rigorous explicit characterization of the discrete networks which give the respective response matrices. Our main result is,

Theorem 2.2. *The response matrix* $\mathbf{W}$ *of any* N*-terminal elastodynamic network of springs and masses with frequency* ω *is a rational function in* $z = 1/\omega^2$ *of the form*

$$\mathbf{W}(z) = \mathbf{A} - \frac{\mathbf{M}}{z} + \sum_{j=1}^{N} \frac{\mathbf{C}_j}{z - z_j}, \tag{2.6}$$

where N *is an arbitrary positive integer,* $\mathbf{M}$ *is the diagonal matrix with the masses of the boundary nodes in the diagonal,* $\mathbf{C}_j$ *is real symmetric negative semidefinite, and the static response* $\mathbf{W}(z = \infty) = \mathbf{A}$ *is real symmetric positive semidefinite and balanced. The poles* z_j *are finite, real and positive. Moreover, for any* N*, any matrix of the form (2.6) can be achieved by a discrete networks of springs and masses with all its nodes in a small neighborhood of the convex hull of the initial set of* N *nodes.*

The study of the characterization of the elastic response for the static case, and the limit to the continuum was carried on by Camar-Eddine and Seppecher in [39] with the help of epi-convergence tools. In the spirit of [39], one of **my next research project** would be the study of the continuum limit of our results obtained for the elstodynamic discrete case, see (2.6), and understand what new equations should be used to model the macroscopic behavior of elastodynamic meta-materials.

Another **interesting research project I plan to consider in the future**, consists in the extension of the program to electromagnetism and acoustics.

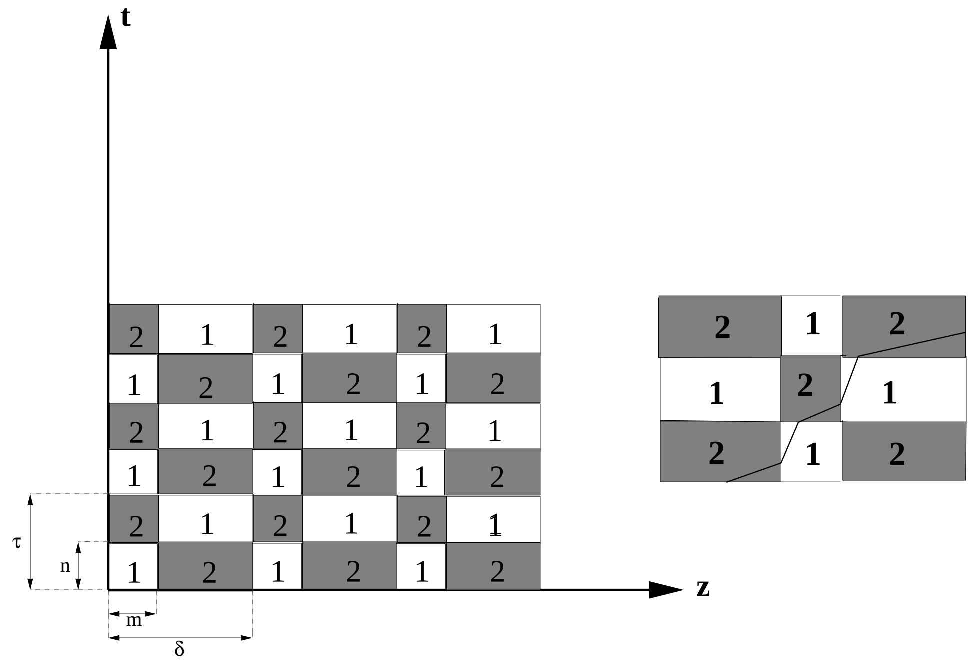

Figure 6: Space-time rectangular microstructure

2.3 Wave propagation through dynamic materials

Another part of my current research is related to the time-space homogenization, i.e., homogenization of problems with fast oscillating periodic coefficients in time and space. In particular, in a joint work with Konstantin Lurie and Suzanne Weekes (see [22]), we studied the wave propagation through a spatio-temporal material composite with rectangular microstructure. The material has a doubly periodic structure with rectangular micro-geometry in one spatial dimension and time. Both spatial and temporal periods in this dynamic material are assumed to be the same order of magnitude. In [22] we studied the standard wave equation with variable coefficients, i.e.,

$$(\rho u_t)_t - (k u_z)_z = 0$$

We consider a checkerboard micro-geometry where variables cannot be separated. The rectangles in a space-time checkerboard are assumed filled with materials differing in the values of phase speeds $\sqrt{\frac{k}{\rho}}$ but having equal wave impedance $\sqrt{k\rho}$. The grid of a checkerboard structure with given δ and τ is defined by horizontal lines $t = i\tau, t = n + i\tau$ and vertical lines $z = i\delta, z = m + i\delta$ for $i \in \mathbb{N}$, as illustrated in Figure 6.

The big difference of the dynamic materials when compared to the classical static composite, is that in the former case the design variables will be time dependent. Many applications of dynamic materials including some unusual effects demonstrated by them, are described in [23] and reference therein.

Within certain parameter ranges, there was numerically observed in [23], the formation of distinct and stable limiting characteristic paths, i.e., limit cycles, that attract neighboring characteristic after a few time periods. The average speed of propagation along the limit cycles remains the same throughout certain ranges of parameters of the micro-geometry, and was called in [23] a plateau effect. Based on numerical evidence it was conjectured [23] that a checkerboard structure is on a plateau if and only if it yields stable limit cycles.

As it has been shown in [24] a dynamic disturbance on a scale much greater than the scale of spatio-temporal microstructure may perceive this formation as a new material with its own effective properties. This is certainly true when the energy carried by the waves is bounded.

On the other hand, there are spatio-temporal assemblages that may not demonstrate this property; specifically in [23] it was observed that the **energy in a checkerboard may, for certain parameter ranges, exhibit exponential growth in time**. This phenomena came up as a consequence of some special kinematics of characteristics, such that the energy is periodically pumped into the wave as it travels through the checkerboard. This happens each time the characteristics (shown in the left top corner of Figure 6 as broken lines) enter the material with higher phase velocity via the horizontal interface. The energy growth then appears to be exponential, with the exponent proportional to the logarithm of the ratio of higher/lower phase velocity.

It is certainly important to better specify conditions that lead up to such growth. In [22] we prove the conjecture that a checkerboard structure is on a plateau if and only if it yields stable limit cycles and we give a detailed analytic characterization of the plateau effect. One of our main results is,

Proposition 2.3. *A structure yields two limit cycles, one stable and the other unstable, if and only if the structure is on a plateau, i.e., the following two pairs of inequalities hold simultaneously:*

$$\begin{cases} \dfrac{a_1\tau + \left(1 - \dfrac{a_1}{a_2}\right) m - \delta}{a_1 - a_2} \leq n \leq \dfrac{a_1\tau + \left(1 - \dfrac{a_2}{a_1}\right) m - \delta}{a_1 - a_2}, \\ \dfrac{m - a_2\tau + \dfrac{a_2}{a_1}(\delta - m)}{a_1 - a_2} \leq n \leq \dfrac{m - a_2\tau + \dfrac{a_1}{a_2}(\delta - m)}{a_1 - a_2}. \end{cases} \tag{2.7}$$

where a_1 and a_2 are the two phase speeds in material 1 and material 2, respectively.

Our result stated in Proposition 2.3, completely characterizes the limit cycles with average speed equal to 1, which is very important because these limit cycles are related with the **energy accumulation phenomena**, as showed in [23].

The results obtained by us are a first step towards the understanding of energy accumulation phenomena for wave propagation through dynamic materials, which may lead to important applications, one of them being the possibility to develop new efficient devices to focus and concentrate energy (i.e., laser knife).

Among our **future projects** I will mention:

- the extension of our analysis to the 2-D and 3-D wave equations, and also to different micro-geometries in time-space

- necessary and sufficient conditions on the material parameters (depending on the micro-geometry), under which the energy will stay bounded in time. This in turn will lead to the question of effective properties for these dynamic composites

- interdepartmental collaborations with experimentalists and industry researchers in an effort to build such dynamic materials, thus having also the possibility to test our analytical results and gain more intuition about the phenomena we study.

2.4 Inverse problems in electromagnetism

In the previous academical year, from my interactions with professor Graeme W. Milton at University of Utah, I became involved in a new research project regarding the identification of heterogeneities for electromagnetism in lossy inhomogeneous media, for the non-monochromatic case based only on boundary measurements. The term lossy means that the time-averaged dissipation of mechanical or electrical energy into heat is positive everywhere, which imply that, for example in the context of electromagnetism, the permittivity tensor is complex with a positive definite imaginary part. An immediate application of such a result will be in the context of impedance tomography.

In the quasistatic regime the constitutive relation between the fields D with $\nabla \cdot D = 0$ and $E = -\nabla V$ in the time domain is,

$$D_t = (\boldsymbol{\varepsilon} E)_t + \boldsymbol{\sigma} E \tag{2.8}$$

where by $(\cdot)_t$ we denoted the time differentiation and where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are space dependent and denote the permittivity and conductivity tensors, respectively.

One way to gain some insight about inside heterogeneities from boundary measurements is to formulate suitable minimization principles for the respective model. In the context of variational principles, many authors tried to answer this question previously, [4], [5], [1], [2]. To mention a classical example, we consider the equations of electrostatics in a domain Ω containing a locally isotropic material with scalar real and positive dielectric constant ε, i.e.,

$$\mathbf{D} = \varepsilon \mathbf{E},\ \mathbf{E} = \nabla V,\ \nabla \cdot \mathbf{D} = 0 \tag{2.9}$$

Two energy minimization principles are known for problem (2.9), the Dirichlet principle and its dual, the Thompson principle. A very important property of these principles is that their corresponding minimum energies are expressible in terms of boundary values of the fields. Thus these two minimization principles provide bounds for the Dirichlet to Neumann map associated to the problem (2.9), and even more, as observed by Berryman and Kohn in [3], they can also be used to obtain information about the permittivity distribution inside the domain based only on knowledge of the two quadratic functionals associated to the Dirichlet to Neumann map and its inverse, respectively.

Building on the quasistatic case modelled by (2.8), one can consider the following mathematical model for the case when the modulli are time and space dependent, i.e.,

$$D_t(t,x) = (\boldsymbol{\varepsilon}(t,x)\nabla V(t,x))_t + \boldsymbol{\sigma}(t,x)\nabla V(t,x) \tag{2.10}$$

The main challenge for the dynamic model (2.10) is that when trying to adapt to it the approaches used previously for the monochromatic case one always ends up with a trivial principle, i.e., a principle which does not help one gain any information about the modulli inside.

So, one of my **current projects** is the implementation of an inverse problem approach for our question, i.e, based on the knowledge of the Dirichlet to Neumann map associated to the model (2.10) to obtain information about the modulli $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ inside. This is a very ill-posed problem and the question is what can one say about the inside modulli based on a finite number of measurements at the boundary.

3 Past research

3.1 Error estimates for periodic homogenization and applications

The last part of my Ph.D. dissertation, was focused on the problem of finding error estimates for the homogenization of elliptic problems in the general case of nonsmooth coefficients. This was an open question even for the classical problem of homogenization. Recently, in a joint work with Bogdan Vernescu (see [25]) we considered the problem,

$$\begin{cases} -\nabla\cdot(A(\frac{x}{\epsilon})\nabla u^{\epsilon}(x)) = f & \text{in } \Omega \\ u^{\epsilon} = 0 & \text{on } \partial\Omega \end{cases}$$

where $A \in L^{\infty}(Y)^{N\times N}$ satisfies the classical ellipticity conditions, is symmetric and Y-periodic, $Y =]0,1[^N$, $\Omega \in \mathbf{R}^N$, smooth convex bounded domain. It is well known that,

$$u^{\epsilon} \rightharpoonup u_0 \ \text{ in } H^1(\Omega)$$

and u_0 verifies

$$\begin{cases} -\nabla\cdot(\mathcal{A}_0\nabla u_0(x)) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

with $\mathcal{A}_0$ is a constant operator described with the help of solutions of the following local problem

$$-\nabla_y\cdot(A(y)(\nabla\chi_j + e_j)) = 0$$

where $\{e_j\}_j$ for the canonical base in $\mathbb{R}^N$. The following error estimate is classical ([26] and references therein),

$$||u^{\epsilon}(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}$$

In [28], using the Periodic Unfolding, Griso proved the above estimate for general L^{∞} coefficients and minimal assumptions on χ_j or u_0. Many authors tried to improve the order of the estimate (3.1) by using suitable boundary layer correctors, defined as solutions to

$$-\nabla\cdot(A(\frac{x}{\epsilon})\nabla\theta_{\epsilon}) = 0 \ \text{ in } \ \Omega \ \ , \ \ \theta_{\epsilon} = \chi_j(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_j} \ \text{ on } \ \partial\Omega$$

Assuming $A \in C^\infty(Y)$, and a sufficiently smooth homogenized solution u_0, it was well known that the corrected series, $u_0(\cdot) + \epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j} - \epsilon\theta_\epsilon(\cdot)$, will approximate the solution u^ϵ of order ϵ in the H^1_0 norm and of order ϵ^2 in the L^2 norm.

The main problem with the above estimate is that they hold with restrictive assumtion on either the coefficient matrix A or on the domain Ω. From the point of view of composite materials, the coefficient matrix can have discontinuities and also if one wants to work in a general convex domain one can't expect more than H^2 regularity for u_0. In this regard, the main question is,

Can one obtain estimates for u^ϵ solution of (3.1) by only assuming bounded coefficients and basic regularity of the homogenized solution u_0, say $u_0 \in H^2$?

There are many papers trying to answer this question, but none of them propose an error estimate result good for general composite media in convex domains!

In [25] we developed a new method which helped us to obtain error estimates for u^ϵ in H^1 norm without assuming any smoothness condition on u_0 or on χ_j. We prove that, for dimension $N \in \{2, 3\}$ we have,

$$||u^\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j}) + \epsilon\beta_\epsilon(\cdot)||_{H^1_0(\Omega)} \leq C\epsilon||u_0||_{H^2(\Omega)}$$

where β_ϵ satisfies

$$-\nabla\cdot(A(\frac{x}{\epsilon})\nabla\beta_\epsilon) = 0 \text{ in } \Omega \ , \ \beta_\epsilon = \chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j}) \text{ on } \partial\Omega$$

and where Q_ϵ is a smoothing operator, explicitly constructed with the help of local averages on cells ϵY.

A **future research project** will be focused on improving the order of the above estimate and extending these results to other nonsmooth elliptic problems. In order to do this, one may consider the second order boundary layer corrector defined as solution to,

$$-\nabla\cdot(A(\frac{x}{\epsilon})\nabla\varphi_\epsilon) = 0 \text{ in } \Omega \ , \ \varphi_\epsilon = \chi_{ij}(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j} \text{ on } \partial\Omega$$

where $\chi_{ij} \in H^1_{per}(Y)$, have zero average and satisfy well posed cell problems. By analyzing the limit behavior of φ_ϵ we prove that, assuming only that $u_0 \in H^3$ and $\chi_j, \chi_{ij} \in H^{1,p}_{per}(Y)$ for some $p > N$, (which, in two dimensions is natural consequence of a Meyer's type regularity result), one can prove an order $O(\epsilon^{\frac{3}{2}})$ error estimate for the H^1 norm of the difference between u^ϵ and the multi-scale expansion series up to the second order.

3.2 Multi-scale analysis of PDE's with variable coefficients and variable geometries

A second part of my Ph.D. dissertation research was concentrated on the homogenization theory for elliptic equations with variable coefficients and variable geometries.

While visiting University Paris VI, the summers of 2004 and 2005, together with the Homogenization group at Laboratoire J. L. Lions, we started working on the asymptotic analysis of linear elliptic problems in perforated domains. In our joint paper [30], we approached these problems building on the periodic unfolding method introduced in [29]. We presented a few of the classical results in a new light, and developed a new method for the multi-scale analysis of phenomena arising on surfaces and thin layers (see [32]), which has produced very interesting new results, such as the limit analysis of the thick Neumann Sieve with variable coefficients for the case when the sieve does not have a uniform thickness.

The thin/thick Neumann sieve problem consist of solving an elliptical PDE on a domain composed of two parts separated by a perforated hyperplane/3D-thin layer, with Dirichlet data on the exterior boundary, continuity conditions for the solution across the perforations and Neumann data proportional with the jump across the remaining part of the hyperplane.

Since the pioneering work of Cioranescu and Murat [31], these problems have been studied by many authors during the last 30 years but there are no computationally efficient error estimates for the homogenization process. In this regard, the advantage of the unfolding method is that, in the periodical setting, it extremely simplifies the existing proofs and the new formulation of the limit problem allows one to obtain very interesting error estimates, even for the case with nonsmooth coefficients (see [28], [25]) The method has four fundamental steps:

1. Definition of one or more suitable unfolding operators depending on the geometry of the problem

2. Finding the exact L^p-bounds for the unfolding operators

3. Defining the proper test functions in order to capture the contribution of the particular geometry to the limit problem, e.g., potential type test function for the perforated domains.

4. Passing to the limit to obtain the unfolding formulation for the limit problem.

One of the main properties of the unfolding operator is that it replaces, integrals on Ω by integrals on the product space $\Omega \times Y$ (where Y is the periodicity cell), and weak convergence by strong convergence, which makes it very clear and easy to manipulate.

Another part of my research in multi-scale analysis was concentrated on spectral homogenization. In a joint work with Bogdan Vernescu [33], we studied the homogenization of the Steklov spectral problem associated to linear or nonlinear operators on the Neumann Sieve model. Using G-convergence results we were able to develop a general technique for the asymptotic analysis of such problems associated to the Laplace operator.

The method developed by us in [33], was later applied in [34] to obtain the limit problem of a Steklov problem associated to the linear elasticity operator on the Neumann sieve. This problem appeared in the context of the earthquake initiation model described below in Section 3.3, and the limit analysis of the first eigenvalue provided interesting information about the stability of the minimum for $\mathcal{W}_\epsilon$ defined at (3.1) below.

Another interesting observation is that the limit analysis of the Steklov problems for the Laplace operator on the Neumann sieve is equivalent with the description of the asymptotic behavior for the spectra of the DtN map associated to the Neumann Sieve.

3.3 Multi-scale phenomena in earthquake modelling

A third part of my Ph.D. dissertation research was dedicated to the analysis of the multi-scale phenomena appearing in the earthquake initiation. In a joint work with Bogdan Vernescu (WPI), and Ioan Ionescu (Savoie, France) [34], we considered the three dimensional shearing of an elastic domain which contains an internal boundary (the fault) located on a plane (the fault plane). The contact on the fault is described through a slip weakening friction (i.e. the friction force decreases with the slip). This friction law is used in the geophysical context of earthquakes modelling and experimental studies. The geometry of the model is represented by an open and bounded domain $\Omega \subset \mathbf{R}^3$ cut in two by the hyperplane $\Pi = \{x_3 = 0\}$.

Here by Ω_+ we denoted the part of Ω above the plane Π and Γ_d denotes the exterior boundary of Ω_+. On each square of an ϵ-lattice constructed on the plane Π a 2-dimensional set (**barrier**) of diameter $\delta\epsilon$ where $\delta \doteq \delta(\epsilon) < 1$, is considered. The term **barrier** denotes here a patch on the fault plane where no slip occurs.

Let $\Sigma_0 = \Pi \cap \Omega$ and denote by Γ_t^ϵ the union of all the **barriers** inside Ω. On the fault outside the barriers, i.e., on $\Gamma_f^\epsilon = \Sigma_0 \setminus \Gamma_t^\epsilon$, we consider a slip-weakening type friction law.

Using symmetry arguments, one can associate a minimization problem for the energy functional, i.e.,

$$\mathcal{W}_\epsilon(v) = \frac{1}{2}\|v\|^2_{V_\epsilon} + \int_{\Sigma_0} SH(|v_\tau|) - f(v), \quad \forall v \in V_\epsilon, \tag{3.1}$$

where $f(v) = -\int_{\Sigma_0} \tau^\infty \cdot v_\tau$, H is the antiderivative of the friction coefficient, and V_ϵ being the natural functional space associated with the functional, i.e., functions in H^1 with homogenous Dirichlet on Γ_d and zero tangential component on the barriers.

The macroscopic behavior of a fault with small-scale heterogeneity of rupture resistance (small scale barriers) is difficult to relate to the local properties of the fault. A formal measure of the friction on the fault itself would just be a local particular law, that would be varying with the position along the fault. The problem was then to find a homogeneous friction law as a good replacement of the local friction law. In the geophysical context the problem was studied in two dimensions (anti-plane geometry) to obtain the recalling of the weakening rate through a spectral analysis.

We use Γ-convergence to obtain the limit functional of the sequence $\mathcal{W}_\epsilon$. Our approach is based on a "separation" lemma which is designed to isolate the contribution of the perforations in the limit process. This idea helps us to prove Γ-liminf inequality and offers an ingenuous way to construct the necessary recovery sequence for the Γ-limsup inequality. Through the Γ-limit of the sequence $\mathcal{W}_\epsilon$ defined at (3.1) we propose an equivalent friction law used on a homogeneous fault as a good replacement for the local friction law on the heterogeneous fault.

Our main result is:

Theorem 3.1. *Let* $0 \leq c = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$. *Then the sequence of functionals*

$$\mathcal{W}_\epsilon \;:\; V_\epsilon \subset V \to \mathbb{R}, \;\; \textit{with} \;\; \mathcal{W}_\epsilon(v) = \frac{1}{2} \parallel v \parallel_V + \int_{\Sigma_0} SH(|v_\tau|) - f(v)$$

Γ*-converge with respect to the weak topology of* V *to,* $\overline{\mathcal{W}} : V \to \mathbb{R}$ *with*

$$\overline{\mathcal{W}}(v) = \frac{1}{2} \parallel v \parallel_V^2 + \int_{\Sigma_0} SH(|v_\tau|) - f(v) + \frac{1}{2} c \sum_{i,j=1}^{3} \int_{\Sigma_0} C_{ij} v_i v_j$$

where V *is the natural limit functional space, i.e, functions in* H^1 *and with homogeneous boundary data on* Γ_d*, and* C *is a constant matrix described in [34] with the help of a class of cell problems.*

A brief physical interpretations of this result leads us to the following conclusions:
i) if the barriers are too large (i.e. $c = \infty$) then the fault is locked (no slip)
ii) if $c > 0$ then the fault behaves as a fault under a slip-dependent friction. The slip weakening rate of the equivalent fault is smaller then undisturbed fault. Since the limit slip weakening rate may be negative a slip-hardening effect can also be expected.
iii) if the barriers are too small (i.e. $c = 0$) then the presence of the barriers does not affect the friction law on the limit fault.

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