Coulomb's Law

\[ \vec{F}_{ab} = \frac{1}{4\pi \varepsilon_0} \frac{Q_a Q_b}{r^2} \]

\[ \varepsilon_0 = 8.85 \times 10^{-12} \text{ Farad/meter} \]

Electrostatic field intensity

Interaction between two point charges \( Q_a, Q_b \) same as the interaction between the field \( \vec{E}_a \) and \( Q_b \) and vice versa.

\( \vec{E} \) is defined as the force/unit charge exerted on a test charge \( Q \) in the field.

\[ \vec{E}_0 = \frac{\vec{F}_{ab}}{Q_b} \quad \text{the field of a point charge} \]

Interaction in vacuum \( Q_a \)
If the electric field $E$ is produced by a charge distribution which might change by the presence of the finite test charge $q'$, then

$$E = E_0 + \frac{F}{2q_0} q'.$$

**Principle of Superposition**

A field produced by more than one charge distribution is the vector sum of all the individual field intensities.

**The Electrostatic Potential**

The work required to move a test point charge $q'$ in the electric field with intensity $E$ between $P_1$ and $P_2$ is

$$W = - \int_{P_1}^{P_2} E \cdot q' \, d\xi.$$

The minus sign shows the fact that the work is done against the field, $E$.!
For a closed path
\[ W = -\oint \bar{E} \cdot d\bar{e} \]

For a field produced by a point charge \( q \)
\[ W = -\oint \bar{E} \cdot d\bar{e} = -\frac{q q'}{4\pi \varepsilon_0} \int \frac{d(\bar{r})}{\bar{r}^2} \]

\( \bar{r}_1 \cdot d\bar{e} = d\bar{r} \implies W = -\frac{q q'}{4\pi \varepsilon_0} \int d(\frac{1}{\bar{r}}) = 0 \)

So, in general, by using superposition, \( \oint \bar{E} \cdot d\bar{e} = 0 \) for any field intensity.

This means that the electrical field \( \bar{E} \) is conservative. Then from Stokes' theorem,

\[ \nabla \times \bar{E} = 0 \implies \bar{E} = -\nabla V \]

When \( V \) is a scalar point function.

The minus sign in (1) is because the fields points in the direction of decrease in potential.
\[ \mathbf{E} \cdot d\mathbf{e} = -\nabla V \cdot d\mathbf{e} = -dV. \quad (2) \]

So,

\[ V_2 - V_1 = -\int_{1}^{2} \mathbf{E} \cdot d\mathbf{e} \]

The potential \( V \) at the point \( r \) is given by

\[ V = \int_{r}^{\infty} \mathbf{E} \cdot d\mathbf{e} \]

when the point at infinity was assumed by convention to be zero.

Problems with this convention:

- When the charge is assumed to extend to infinity.

The work \( W \) required to move a point charge \( Q' \) from a point with zero potential \( V = 0 \) is given by

\[ W = V \cdot Q' \Rightarrow V = \frac{W}{Q'} \]

\[ V \text{ --- the work per unit charge.} \]

The points in space with equal potential form the equipotential surfaces. The vector field \( \mathbf{E} \) is called the

\[ \text{--- perpendicular to equipotential surfaces.} \]
Potential produced by a continuous charge distribution.

Density \( \rho = \frac{\Delta q}{\Delta z} \) - coulombs/meter\(^3\).

When \( \Delta q \) is the total charge within the infinitesimal \( \Delta z \).

\( \Delta z \) - small enough but large enough so that the fluctuations of \( \Delta q \) with time or those from one \( \Delta z \) to a neighbouring one are negligible.

Thus \( \rho \Delta z \) contributes an element of potential \( dV \) at a point \( P \) outside \( \Delta z \):

\[
\frac{dV}{4\pi \varepsilon_0} = \frac{\rho \Delta z}{4\pi \varepsilon_0 \Delta z} \rightarrow \frac{dV}{4\pi \varepsilon_0} \text{ potential} = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho \Delta z}{r} dV
\]

Gauss's Law:

Point charge \( Q \) located inside closed \( S \).

The flux of \( \mathbf{E} \) through the element of area, \( da \):

\[
\mathbf{E} \cdot da = \frac{Q}{4\pi \varepsilon_0} \frac{\pi r^2 da}{\pi^2} = \frac{Q}{4\pi \varepsilon_0} \frac{da'}{r^2} = \frac{Q}{4\pi \varepsilon_0} d\Omega
\]

\( da' \) - projection of \( da \) on a plane orthogonal to \( \mathbf{r} \).

\( d\Omega = \frac{da'}{r^2} \) is the element of solid angle at \( P \).
The element of solid angle $d\Omega$ subtended at a point $P$ by an element of surface area $dA$.

$$d\Omega = \frac{dA}{r^2} \quad \text{steradians}$$

\[ \int d\Omega \]

Total flux of electric field through the surface $S$,

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\varepsilon_0} \quad \text{when} \quad (3)$$

The total solid angle subtended by the surface $S$ which completely surrounds $P$ at $P$ is $4\pi$.

Important: Formula (3) is true for a closed surface surrounding a point charge $Q$ located at a point $P$.

If $P$ is located outside the surface than the total angle subtended by the surface at $P$ would be zero.

\[ \oint \mathbf{E} \cdot d\mathbf{a} \]

\[ \oint \mathbf{E} \cdot d\mathbf{a} \]
Total flux of $\mathbf{E}$ (due to more than one point charge located within $S$) is the algebraic sum of the fluxes of each individual charge.

Continuous distributed charge within $S$

Total enclosed charge
$$Q = \oint_{S} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int_{V_{\text{Vol}}} \mathbf{p} \cdot d\mathbf{V}$$

$$\nabla \cdot \mathbf{E} = \frac{\mathbf{p}}{\varepsilon_0}$$

Gauss's Law

Mean Value Theorem
$$\bar{V} = \frac{1}{4\pi R^2} \int_{S} \nabla V \cdot d\mathbf{a} = \text{the average potential over the sphere } S \text{ of radius } R \text{ and center } O.$$

If $\mathbf{p} = 0$ in a region of space including $S$, then
$$\bar{V} = V(0),$$
where $O$ is the center of $S$. 
\[ \nabla = \frac{1}{4\pi} \int_S \nabla \cdot \mathbf{E} \, d\mathbf{S} = \frac{1}{4\pi} \int_S \mathbf{E} \cdot d\mathbf{S} \]

So \[ \frac{\partial \nabla}{\partial t} = \frac{1}{4\pi} \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} = \frac{1}{4\pi} \int_S \mathbf{E} \cdot \frac{\partial d\mathbf{S}}{\partial t} = 0 \]

because \( \mathbf{E} \cdot d\mathbf{S} \) is the flux of \( \mathbf{E} \) through the surface \( S \), and by the assumption that \( \mathbf{E} = 0 \) and Gauss's law should be zero.

So \( \nabla (\mathbf{E}) = \nabla (\vec{0}) = \vec{0} \)

Poisson's and Laplace equations

Conservative field \( \mathbf{E} \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = \Delta \mathbf{E} = 0 \)

Gauss's law

Poisson's equation

If there is a charge free region.

\( \Delta \mathbf{E} = 0 \quad \Rightarrow \quad \text{Laplace equation} \)

Conductors

A conductor defined as a body in which charges can flow freely.

Conductors in Electrostatics \( \Rightarrow \) No flow of charges within the conductor.
Thus \( E = 0 \) within a conductor. In electrostatics, the potential \( V \) must be the same throughout a conductor.

Gauss's law \( \nabla \cdot E = \rho / \varepsilon_0 \) implies that \( \rho / \varepsilon_0 = 0 \) within a conductor. Since charge resides only on the surface of a conductor, the electric field intensity is normal to the surface of the conductor because of the assumption of no flow of charge.

Gauss's law on the box with height \( s \) around the boundary gives

\[
E = \frac{s \sigma}{2 \varepsilon_0}
\]

just outside the boundary of a conductor.

When \( \sigma \) is the surface charge density, induced charges in a hollow conductor

Any charge interior to a hollow conductor will induce a net charge equal in magnitude and opposite in sign so that the net charge must be zero from Gauss's law.
Examples of particular uniform charge distribution

1. Uniform Spherical Charge Distribution. (Sphere of center O, radius R)

\[ E = \frac{1}{4\pi \varepsilon_0} \frac{Q_1}{r^2} \]  \quad \text{for } P \text{ outside the sphere and with } \overline{OP} = z

2. Total charge, \( Q = \frac{4}{3} \pi R^3 \rho \)

\[ E = \frac{\rho_2}{3\varepsilon_0} \]  \quad \text{for } P^1 \text{ inside the sphere with } \overline{OP} = z \text{ and } \rho_2 \text{ is the charge density.}

2. Infinite sheet of charge

\[ E = \frac{\sigma}{2\varepsilon_0} \]  \quad \text{where } \sigma \text{ is the surface charge density.}
3. Conducting plate with a surface charge density $\sigma$.

\[ E = \frac{\sigma}{\varepsilon_0} \]

The Dipole

$S \ll \lambda$

\[ V = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{\lambda_b} - \frac{1}{\lambda_a} \right) \]

Cosine law

\[ \lambda_a = \lambda^2 + \left( \frac{S}{2} \right)^2 + \lambda S \cos \theta \]

\[ \frac{\lambda}{\lambda_a} = \frac{1}{\sqrt{1 + \left( \frac{S}{2\lambda} \right)^2 + \frac{S}{\lambda} \cos \theta}} \]

if $S \ll \lambda$ then

\[ V = \frac{Q S}{4\pi \varepsilon_0 \lambda^2} \cos \theta \]

Dipole moment $\vec{p} = QS$ vector with magnitude $QS$

and directed from $Q_-$ to $Q_+$

Then

\[ V = \frac{\vec{p} \cdot \vec{r}}{4\pi \varepsilon_0 \lambda^2} = \frac{\vec{p}}{4\pi \varepsilon_0} \cdot \nabla \left( \frac{1}{\lambda} \right) \]

\[ E_n = -\frac{\partial V}{\partial n} = \frac{2\vec{p}}{4\pi \varepsilon_0 \lambda^2} \cos \theta \]

\[ E_\theta = -\frac{\partial V}{\partial \theta} = \frac{\vec{p}}{4\pi \varepsilon_0 \lambda^2} \sin \theta, \quad E_\phi = 0 \]
The Linear Quadrupole

\[ V = \frac{1}{4 \pi \varepsilon_0} \left( \frac{Q}{r_a^2} - \frac{2Q}{r} + \frac{Q}{r_b^2} \right) = \]

\[ = \frac{2Qs^2}{4\pi \varepsilon_0 r^3} (3 \cos^2 \theta - 1) \]

for \( r > s \).

The electric field intensity \( E \) for a quadrupole decays as \( \frac{1}{r^4} \) for \( r > s \).

**Multipoles**

1 charge - monopole

dipole - displacement of 1 \( Q \)-charge by a distance \( s \) on the vertical and charge the sign of the new charge.

2 quadrupole - displacement of a dipole, distance \( s \) on the vertical and charge the sign of the 2 new charges.

The 2° pole (multiple) potential decays like

\[ \frac{1}{r^{2+1}} \]
Electrostatic Potential \( V \) due to an arbitrary charge distribution. Nonuniform charge density \( \rho(x, y, z) \)

\[
V = \int_2 \frac{\rho(z)}{4\pi e_0 R} \text{ where } R = (x-x')^2 + (y-y')^2 + (z-z')^2}^{1/2}
\]

\[
\frac{1}{R} = \frac{1}{r} + \left( x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right) \left( \frac{1}{R} \right) + \ldots \quad (1)
\]

where (1) is the Taylor series for \( \frac{1}{R} \) around \( r = 0 \)

Observe that

\[
\left[ \frac{\partial}{\partial x} \left( \frac{1}{R} \right) \right] = \left[ \frac{x - x'}{R^3} \right] = \frac{x}{r^3} = \frac{\theta}{r^2}
\]

where \( \theta = \frac{x}{r} \) is the cosine of the angle between \( \mathbf{r} \) and \( \mathbf{x} \).

\[
V = \int_2 \frac{\rho(z)}{4\pi e_0} + \int_2 \frac{1}{r^2} (x'x + y'y + z'z) \frac{\rho(z)}{4\pi e_0} \quad (2)
\]

where \( x', y', z' \) are the direction cosines.
The first term in (2) is the potential we would have at $P$ if the whole charge were concentrated at the origin.

$$T_1 = \int \frac{1}{r^2} \frac{2 d\mathbf{r}}{2} \quad \text{monopole term}$$

The second term is in fact the dipole moment of a point dipole corresponding to the entire charge distribution.

$$T_2 = \frac{\mathbf{P} \cdot \mathbf{\hat{r}_1}}{4\pi \varepsilon_0 r^2}$$

where

$$\mathbf{P} = \int (\mathbf{x} \times \mathbf{\hat{r}_1} + \mathbf{y} \mathbf{\hat{s}} + \mathbf{z} \mathbf{\hat{k}}) \, 2 d\mathbf{r}$$

$\mathbf{\hat{r}_1}$ is the unit vector along $\mathbf{\hat{r}_1}$ in the direction of $\mathbf{P}$

$$\mathbf{\hat{r}_1} = \mathbf{i} + n \mathbf{j} + m \mathbf{k}$$

$\mathbf{\hat{r}_1}$ is the dipole moment of a charge distribution $\mathbf{P}$

$$\mathbf{P} = \int \mathbf{\hat{r}_1} \, 2 d\mathbf{r}$$

If

$$Q = \int 2 d\mathbf{r}$$

then the dipole moment of the charge distribution is independent of the choice of the origin!!
The potential produced by an arbitrary charge distribution is the same as that if
1) a point charge equal to the net charge of the
distribution.

Plus
2) a point dipole with a dipole moment equal to the dipole moment of the
distribution.

Plus
3) a point quadrupole with the quadrupole moment of the distribution
and so on, all located at the origin.
The average field intensity inside a sphere containing an arbitrary charge distribution.

Suppose first we only have a point charge \( Q \) located within the sphere of radius \( R \).

Assume that \( Q \) is on the \( z \) axis.

Then the average field over the volume of the sphere is in the \( z \) direction.

\[
E_z = \frac{1}{V} \int_E E_z \, dV = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} \int_0^\pi E_z \sin \theta \, r^2 \sin \theta \, d\theta \, d\varphi \, dr
\]

One can separate the integral in 2 parts.

One over the region \( B_0 / B_{\|} \) and one over \( B_{\|} / B_0 \).

The field over \( B_0 / B_{\|} \) is two as it can be showed that while \( E_z \) decreases as the square of the distance from \( Q \), \( dz \) increases as the square of the distance and therefore the product is independent of the distance.

\( R > 1 \Rightarrow dz > \infty \) and at \( dz, E_z < 0 \), so
The two contributions cancel out.

For the field over the inner ball, \( B \), we have

\[
V = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r_1}
\]

potential at \( P(x, y, z) \) \( \approx B \).

where \( r_1 \) is the vector between \( P \) and \( Q \).

Taylor expansion for \( V \) in terms of \( \frac{r_2}{r_1} \) gives.

\[
V = \frac{Q}{4\pi \varepsilon_0} \left[ 1 + \frac{r_2}{r_1} \cos \theta + \ldots \right].
\]

Simple calculations

\( \cos \theta = 2 \) \( \Rightarrow \frac{\partial r}{\partial z} = \cos \theta \)

lead to

\[
\vec{E}_z = \frac{\vec{E}_2 \cdot z}{z} = -\frac{P}{4\pi \varepsilon_0 R^3}
\]

where \( P = Q \lambda_1 \) is the dipole moment of the charge \( Q \).

For an arbitrary charge distribution on the average field

\[
\vec{E}_z = -\frac{P}{4\pi \varepsilon_0 R^3}
\]

when \( P = \int \vec{S} \cdot \vec{r} \, \mathrm{d}z \)
Summary of important results

Dielectric materials in electrostatic fields

All the electrons are bonded to their parent molecules.

The only motion within the dielectric is the displacement of the positive and negative charges of the molecules in opposite directions.

So, dielectric in get polarized.

Electrical field

The molecules in the dielectric will then have an induced dipole moment \( \mathbf{p} \), and these dipole moments will produce a fixed field which adds up to the external field.
The electric polarization

\[ P - \text{ dipole moment / unit volume.} \]
\[ N - \text{ number of molecules / unit volume.} \]

\[ P = N \cdot p \]

Unit volume – small enough volume.

Electrostatic field

\[ E = \frac{1}{4\pi \varepsilon_0} \sum \frac{\rho'_d}{r^2} + \frac{1}{4\pi \varepsilon_0} \sum \frac{\sigma'_d}{r^2} \]

\[ \sigma' = \rho \cdot n \]
\[ \rho' = -\nabla \cdot P. \]

Bound charges within dielectric.

The local field intensity

\[ E_{loc} - \text{ time and space average of total electric field intensity acting on a particular molecule.} \]

\[ E_{loc} \sim E + \frac{P}{3\varepsilon_0}. \]

( in general \[ \varepsilon_{loc} = E + 6 \frac{P}{5\varepsilon_0} \), \(6 \text{ constant}. \)
The displacement vector $\mathbf{D}$

\[ \int_{S} \mathbf{E} \cdot d\mathbf{a} = \int_{V} \nabla \cdot \mathbf{E} \, dV = \frac{Q}{\varepsilon_{0}}. \quad \text{Gauss's law}. \]

For dielectrics,

\[ Q = \int_{S'} (\mathbf{P} + \mathbf{P}') \, d\mathbf{a} \]

This includes $\mathbf{P}'$, as it can be assumed to extend over a small thickness and thus embedded in $S'$. So Gauss's law for dielectrics:

\[ \nabla \cdot \mathbf{E} = \frac{\mathbf{P} + \mathbf{P}'}{\varepsilon_{0}} \quad \Rightarrow \quad \nabla \cdot (\varepsilon_{0} \mathbf{E} + \mathbf{P}) = \mathbf{P}. \]

But \[ \mathbf{P}' = -\nabla \cdot \mathbf{P} \]

Electric Displacement

\[ \mathbf{D} = \varepsilon_{0} \mathbf{E} + \mathbf{P} \]

We have

\[ \nabla \cdot \mathbf{D} = \mathbf{P} - \text{free charge density}. \]

\[ \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_{0}} - \frac{\mathbf{P}}{\varepsilon_{0}}. \]
The electric susceptibility $\chi_e$ of homogeneous linear and isotropic materials is given by:

$$\mathbf{P} = \chi_e \mathbf{E} = \chi_e (\mathbf{E} + \mathbf{b} \frac{\mathbf{P}}{\varepsilon_0})$$

where $\chi_e$ is the molecular polarizability, $E$ is the electric field, and $N$ is the number of molecules per unit volume.

So

$$\mathbf{P} = N \chi_e \mathbf{E}$$

where

$$\chi_e = \frac{N \chi}{\varepsilon_0 - N \chi}$$

So

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \chi_e \varepsilon_0 \mathbf{E}$$

where $K_e = 1 + \chi_e$ is the dielectric constant.

Polar molecules

Molecules having permanent dipole moments align themselves with the electric field and have a susceptibility that is inversely proportional with the temperature.
Linear, homogeneous, isotropic dielectric.

$$\varepsilon' = \left( \frac{k_{e} - 1}{k_{e}} \right) \varepsilon_0$$

If \( \varepsilon = 0 \) \( \Rightarrow \varepsilon' = 0 \) and therefore the only bound charge resides on the surface.

Energy in dielectrics

$$W = \frac{1}{2} \int \rho V dV = \frac{1}{2} \int \nabla \cdot \mathbf{D} dV$$

$$= \frac{1}{2} \varepsilon_0 \int \nabla \cdot \mathbf{D} dV = \frac{1}{2} \varepsilon_0 \int \mathbf{D} \cdot d\mathbf{V} - \frac{1}{2} \varepsilon_0 \int \delta dV = \varepsilon_0 \int \mathbf{D} \cdot d\mathbf{V}$$

$$= \frac{1}{2} \varepsilon_0 \int \mathbf{D} \cdot d\mathbf{V} + \frac{1}{2} \varepsilon_0 \int \mathbf{D} \cdot d\mathbf{E}$$

Surface

$$= \frac{1}{2} \varepsilon_0 \int \mathbf{D} \cdot d\mathbf{E}$$

all you