Dielectrics

Summary of important results

Dielectric materials in electrostatic fields

All the electrons are banned to their parent molecules.
The only motion within the dielectric is the small displacement of the positive and negative charges of the molecules in opposite directions.

So, Dielectric in Electrical field get polarized.

The molecules in the dielectric will then have an induced dipole moment $\bar{P}$, and these dipole moments will produce a field which adds up to the external field

The electric polarization

P — dipole moment / unit volume.

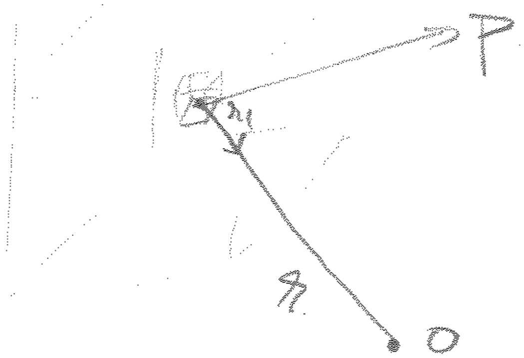
N — number of molecules / unit volume.

$$P = Np$$

Unit volume — τ — small enough volume.

Electrostatic field

$$E = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\rho' d\tau}{r^2}\bar{r}_1 + \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma' da}{r^2}\bar{r}_1$$



$$\begin{cases} \sigma' = P \cdot n \\ \rho' = -\nabla \cdot P \end{cases}$$

bound charges within dielectric.

The local field intensity

$\bar{\mathcal{E}}_{loc}$ — time and space average of total electric field intensity acting on a particular molecule

$$\bar{\mathcal{E}}_{loc} \simeq E + \frac{P}{3\varepsilon_0}. \qquad \left(\begin{array}{l}\text{in general} \\ \bar{\mathcal{E}}_{loc} = E + b\dfrac{P}{\varepsilon_0}, \ b \text{ constant}\end{array}\right)$$

The Displacement vector D

$$\int_S E \cdot da = \int_\tau \nabla \cdot E \, d\tau = \frac{Q}{\varepsilon_0}.$$ — Gauss's law.

For dielectrics.

$$Q = \int_\tau (\rho + \rho') \, d\tau$$

This includes σ', as it can be assumed to extend over a small thickness and thus embedded in ρ'.

So Gauss's law for dielectric.

$$\left.\begin{aligned} \nabla \cdot E &= \frac{\rho + \rho'}{\varepsilon_0} \\ \text{But} \quad \rho' &= -\nabla \cdot P \end{aligned}\right\} \Rightarrow \nabla \cdot (\varepsilon_0 E + P) = \rho.$$

Electric Displacement

$$D = \varepsilon_0 E + P$$

We have

$$\nabla \cdot D = \rho$$ — free charge density.

$$\Downarrow$$

$$E = \frac{D}{\varepsilon_0} - \frac{P}{\varepsilon_0}.$$

The electric Susceptibility χ_e

Linear and isotropic homogeneous materials.

$$P = \alpha \, E_{loc} = \alpha\left(E + b\frac{P}{\varepsilon_0}\right)$$

where α is the molecular polarizability

$$P = Np = N\alpha\left(E + b\frac{P}{\varepsilon_0}\right).$$ where N is the number of molecules/vol.

So

$$\underline{P = \varepsilon_0 \chi_e E}$$ where

$$\chi_e = \frac{N\alpha}{\varepsilon_0 - N\alpha b}$$

So $$D = \varepsilon_0(1+\chi_e)E = K_e \varepsilon_0 E$$ where

$$K_e = 1 + \chi_e$$ is the dielectric constant.

Polar molecules

Molecules having permanent dipole moments align themselves with the electric field and have a susceptibility as inverse proportional with the temperature.

Linear, homogeneous, isotropic dielectric.

$$-\rho' = \left(\frac{K_e - 1}{K_e}\right)\rho.$$

If $\rho = 0 \Rightarrow \rho' = 0$ and therefore the only bound charge resides on the surface.

Energy in dielectrics

$$W = \frac{1}{2}\int_{V} \rho V d\tau = \frac{1}{2}\int_{V} (\nabla\cdot E) d\tau$$

$$= \frac{1}{2}\varepsilon_0 \int_{Vol} \nabla\cdot D V d\tau =$$

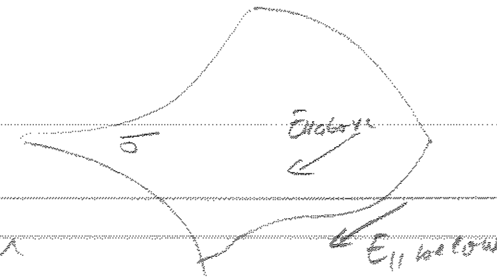
$$= \frac{1}{2}\varepsilon_0 \int_{Vol} \nabla\cdot(DV) - \frac{1}{2}\varepsilon_0 \int_{Vol} D\nabla V =$$

$$= \frac{1}{2}\varepsilon_0 \int_{Surface} DV d\bar{a} + \frac{1}{2}\varepsilon_0 \int_{Vol} D\cdot E$$

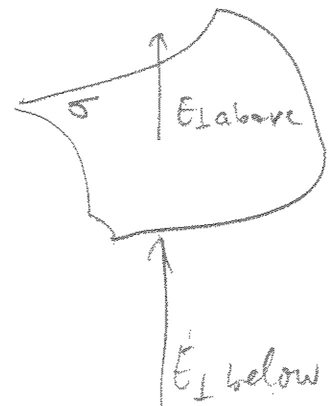
$$= \frac{1}{2}\varepsilon_0 \int_{all\ space} D\cdot E$$

Electrostatic boundary conditions

$$E_{\parallel\,above} = E_{\parallel\,below}$$



$\oint E \cdot dl = 0$ applied to a thin rectangular loop.



$$\bar{E}_{\perp\,above} - \bar{E}_{\perp\,below} = \frac{1}{\varepsilon_0}\sigma \cdot \hat{n}$$

$$\oint E \cdot da = \frac{1}{\varepsilon_0}\sigma A.$$

For potential V

$$\begin{cases} V_{above} = V_{below} \\ \dfrac{\partial V_{above}}{\partial n} - \dfrac{\partial V_{below}}{\partial n} = -\dfrac{1}{\varepsilon_0}\sigma. \end{cases}$$

where.

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{n}.$$

Energy

The work done to move a charge Q to P against an electric E is

$$W = Q \cdot V(P)$$

The E-M in the quasistatic regime

$D = K_e \varepsilon_0 E$

$\nabla \cdot D = 0 \Rightarrow -\nabla \cdot (K_e \varepsilon_0 \nabla V) = \rho_f$ - free charge density

Separation of variable formula in 2D and 3D

- Cartesian coordinates. (Read - Examples 3, 4, 5. Do the plots)

- Spherical coordinates

Assuming azimuthal symmetry (linear homogeneous and isotropic media)

$$\nabla^2 V = 0$$

$$\Updownarrow$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0.$$

Solution

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

When $P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$.

Polar coordinates

$\Delta U = 0$ in D

$U = f$ on ∂D,

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0. \qquad f = \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

Solution of the form

$$U(r,\theta) = R(r)\,\Theta(\theta)$$

$$\begin{cases} \Theta'' + \lambda^2 \Theta = 0 \\ r^2 R'' + rR' - \lambda^2 R = 0 \\ \lambda = n \text{ - integer (periodicity)} \end{cases} \Rightarrow \begin{cases} \Theta(\theta) = a_n e^{in\theta} \\ R(r) = \begin{cases} b_0 + c_0 \log r & (n=0) \\ b_n r^n + c_n r^{-n} & (n \neq 0) \end{cases} \end{cases}$$

$$U(r,\theta) = \sum_{n=-\infty}^{\infty} a_n \left(\frac{r}{a}\right)^{|n|} e^{in\theta}$$

In general solution of

$\Delta U = 0$ is

$$U(r,\theta) = a_0 + b_0 \log r + \sum_{n=1}^{\infty} \left[a_n r^n + b_n r^{-n}\right] e^{in\theta}$$

or

$$U(r,\theta) = a_0 + b_0 \log r + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos(n\theta) + \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) \sin(n\theta)$$

For annulus.

$$U(R_1,\theta)=g_1(\theta)$$

$$U(R_2,\theta)=g_2(\theta)$$

$$\begin{cases} a_0+b_0\ln R_1=\frac{1}{2\pi}\int_0^{2\pi}g_1(s)\,ds. \\ a_0+b_0\ln R_2=\frac{1}{2\pi}\int_0^{2\pi}g_2(s)\,ds. \end{cases}$$

$$\begin{cases} a_nR_1^n+b_nR_1^{-n}=\frac{1}{\pi}\int_0^{2\pi}g_1(s)\cos(ns)\,ds \\ a_nR_2^n+b_nR_2^{-n}=\frac{1}{\pi}\int_0^{2\pi}g_2(s)\cos(ns)\,ds \end{cases}$$

$$\begin{cases} c_nR_1^n+d_nR_1^{-n}=\frac{1}{\pi}\int_0^{2\pi}g_1(s)\sin(ns)\,ds. \\ c_nR_2^n+d_nR_2^{-n}=\frac{1}{\pi}\int_0^{2\pi}g_2(s)\sin(ns)\,ds \end{cases}$$