Math 5110

Midterm 1

30 pts total

Problems:

1. Consider the following model for the populations of two species: the cinnabar moth and ragwort, a perennial plant on which the moth feeds. Because the moth only lives for a year, the model tracks the number of moth eggs (E_n) and the total ragwort biomass (B_n) each year.

$$E_{n+1} = \beta B_n, \quad B_{n+1} = \kappa \exp\left(-\gamma \frac{E_n}{B_n}\right). \tag{1}$$

Remember that the notation $\exp(x)$ denotes the exponential function e^x. The variable E_n is measured in number of eggs, and the variable B_n is measured in tons of ragwort. The parameters β, κ, and γ are all non-negative. This model can be rescaled to non-dimensional form

$$Y_{n+1} = X_n, \quad X_{n+1} = \exp\left(-a \frac{Y_n}{X_n}\right). \tag{2}$$

Here, Y_n representes eggs, and X_n represents ragwort biomass (both in non-dimensional units).

10 pts (a) Perform the rescaling of Equation (1) to get Equation (2). Define X_n, Y_n, and a in terms of the variables and parameters seen in Equation (1).

We will now analyze the non-dimensional version of the model (Equation (2)).

5 pts (b) Find all *positive* equilibria of the model.

5 pts (c) Find the Jacobian matrix and evaluate it at each equilibrium that you found in part (b).

10 pts (d) Find the parameter values for which each positive equilibrium is stable.

Solution:

(a) We know the units of E_n and B_n from the problem statement:

$$[E_n] = \text{ number of eggs}, \quad [B_n] = \text{ tons of ragwort.}$$

From this we can immediately deduce the units of β:

$$[\beta] = \frac{\text{number of eggs}}{\text{tons of ragwort}}.$$

Also, we know that the output of the exponential function must be dimensionless, therefore the units of κ are clear:

$$[\kappa] = \text{ tons of ragwort.}$$

3 points for correctly identifying units

Finally, we know that the input of the exponential function must be dimensionless, this implies that the units of γ must be

$$[\gamma] = \frac{\text{tons of ragwort}}{\text{number of eggs}}.$$

4 points for correct X & Y

Now we must choose the characteristic scales for E_n and B_n. Choosing a scale for B_n is the most obvious, as there is only one parameter with units of "tons of ragwort". We define the non-dimensional amount of ragwort as $X_n = B_n/\kappa$. We now notice that the product $\kappa\beta$ has units of "number of eggs," and we therefore define the non-dimensional amount of moth eggs $Y_n = E_n/(\kappa\beta)$. Substituting into Equation (1), we have

$$\kappa\beta Y_{n+1} = \kappa\beta X_n, \quad \kappa X_{n+1} = \kappa \exp\left(-\gamma\frac{\kappa\beta Y_n}{\kappa X_n}\right).$$

2 pts for algebra

Canceling terms finally gives us

$$Y_{n+1} = X_n, \quad X_{n+1} = \exp\left(-\gamma\beta\frac{Y_n}{X_n}\right).$$

1 pt for a

Defining the non-dimensional parameter $a = \gamma\beta$ gives us the rescaled model in Equation (2).

(b) To find the equilibrium, we solve the equations

$$Y^* = X^*, \quad X^* = \exp\left(-a\frac{Y^*}{X^*}\right).$$

2 pts for set up

The first equation allows us to simplify the second:

3 pts for equil.

$$X^* = \exp(-a).$$

So we have that $\boxed{X^* = Y^* = e^{-a}.}$ This equilibrium is positive for all a.

(c) We calculate the Jacobian matrix (the first term requires the chain rule):

$$J = \begin{pmatrix} \frac{aY}{X^2}\exp\left(-a\frac{Y}{X}\right) & \frac{-a}{X}\exp\left(-a\frac{Y}{X}\right) \\ 1 & 0 \end{pmatrix}.$$

3 pts for general J

Plugging in the equilibrium from part b gives

$$J\left(e^{-a}, e^{-a}\right) = \begin{pmatrix} \frac{a}{e^{-a}}\exp(-a) & \frac{-a}{e^{-a}}\exp(-a) \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -a \\ 1 & 0 \end{pmatrix}.$$

2 pts for evaluation

(d) We calculate the trace $\tau = a$ and determinant $\Delta = a$. We can now check to see when the fixed point is stable. Test 1:

$$\Delta > -1 + \tau$$
$$\Rightarrow a > a - 1, \quad \text{always true.}$$

3 pts for test 1

Test 2:

3 pts for test 2

$$\begin{aligned}\Delta &> -1-\tau\\ \Rightarrow a &> -a-1\\ \Rightarrow 2a &> -1\\ \Rightarrow a &> -\frac{1}{2}.\end{aligned}$$

This is always true because we know (from the problem statement) that $a \geq 0$. Test 3:

$$\begin{aligned}\Delta &< 1\\ \Rightarrow a &< 1.\end{aligned}$$

3 pts for test 3

Therefore we know that for the fixed point to be stable, we must have $a < 1$. If $a > 1$, the fixed point is unstable.

1 pt for final condition

2. Consider the following model of a single population in an environment with limited resources:

$$P_{n+1} = P_n \exp\left(r(1 - \frac{P_n}{K})\right). \tag{3}$$

This equation is sometimes called the discrete form of the continuous logistic model (not to be confused with the discrete logistic model that we saw in class). The parameter K is the carrying capacity of the environment and r is the non-dimensional reproduction of the population. Both are assumed to be positive.

5 pts (a) Find *both* equilibria of the model.

15 pts (b) Determine when each equilibrium is stable.

10 pts (c) A bifurcation occurs when $r = 2$. Determine what type of bifurcation it is and show why.

Solution:

(a) We are looking for equilibria that satisfy

$$P^* = P^* \exp\left(r(1 - \frac{P^*}{K})\right).$$

Clearly $P^* = 0$ is one equilibrium. To find the other, we divide the equation by P^* and have

2 pts

$$1 = \exp\left(r(1 - \frac{P^*}{K})\right).$$

Taking the logarithm, we have

3 pts

$$r\left(1-\frac{P^*}{K}\right)=0.$$

Some algebra gives $P^* = K$ is the other (positive) equilibrium.

(b) If $P_{n+1}=F(P_n)$, we calculate the derivative

$$F'(P)=\exp\left(r(1-\frac{P}{K})\right)-\frac{rP}{K}\exp\left(r(1-\frac{P}{K})\right).$$

5 pts for correct deriv.

Plugging in the zero equilibrium, we have

$$F'(0)=\exp(r)=e^r.$$

2 pts for evaluation

3 pts for analysis

The derivative will be greater than one if $r>0$, which we know is true from the problem statement. Therefore the equilibrium is unstable.

We now evaluate the derivative at the positive equilibrium

$$F'(K)=\exp\left(r(1-\frac{K}{K})\right)-\frac{rK}{K}\exp\left(r(1-\frac{K}{K})\right)=1-r.$$

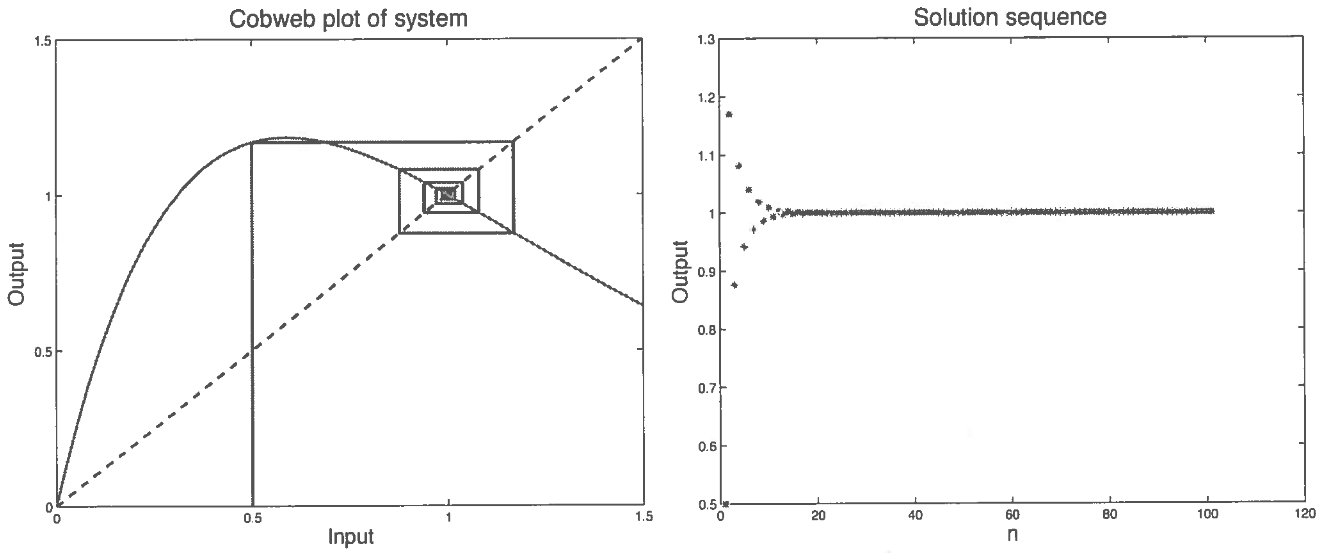
2 pts for evaluation

3 pts for analysis

We know that r is positive, therefore this is always less than 1. The derivitive will be greater than -1, and the equilibrium will be stable for $r<2$. The equilibrium will be unstable for $r>2$.

(c) From the previous section, we know that the positive equilibrium $P^*=K$ has an eigenvalue given by $\lambda=1-r$. At $r=2$, this equilibrium goes from stable to unstable, as the eigenvalue passes through -1 (and is real, obviously). This means that the fixed point underwent a period doubling bifurcation. This is illustrated in the Figures 1 and 2.

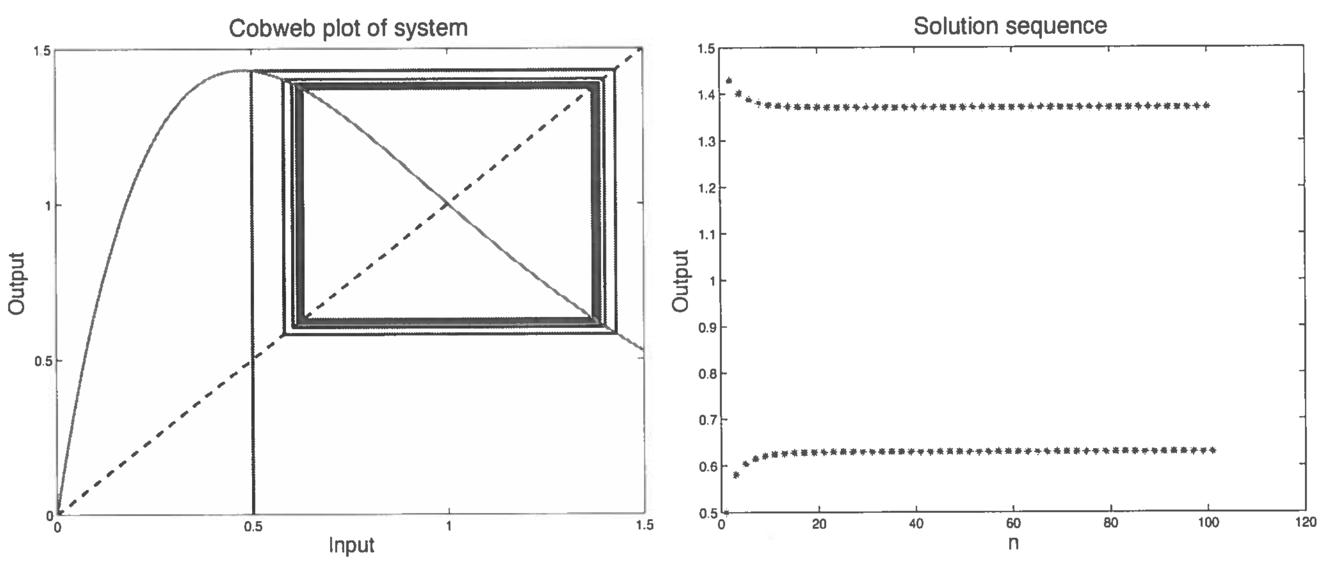
Showing that eigenvalue crosses -1 @ $r=2$ gets you full credit. Showing pictures before & after bifurcation is good enough for partial credit.



(a) Cobweb plot of population.

(b) Time evolution of population.

Figure 1: Behavior of the model for $r = 1.7$; before the bifurcation.



(a) Cobweb plot of population.

(b) Time evolution of population.

Figure 2: Behavior of the model for $r = 2.1$; after the flip bifurcation.