RIEMANN-ROCH THEOREM FOR CURVES

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ABSTRACT. This paper aims to provide an elementary proof of Riemann-Roch theorem for curves after providing an introduction to a number of basic concepts found in algebraic geometry and complex analysis. The Riemann-Roch theorem is a powerful tool for relating a purely topological invariant to an alternative algebraic setting for the same object and has applications in many fields of math.

CONTENTS

1. INTRODUCTION

The Riemann-Roch theorem is a fundamental tool in algebraic geometry. Its usefulness includes but is not limited to classifying algebraic curves according to useful topological invariant and determining whether prescribed poles and zeros exist in the spectrum of functions in the field of fractions of a certain curve.

Using elementary machinery to analyze functions on a curve, we will prove the following theorem:

Theorem 1 (Riemann-Roch). *Let C be a smooth curve, let K_C denote the canonical divisor on C. Then there exists an integer $g \geq 0$ such that for every divisor $D \in Div(C)$ the following equality holds*

$$\ell(D) - \ell(K_C - D) = \deg D - g + 1$$

Here, g is the genus of the curve C.

2. DEFINITIONS AND NOTATION

In this section, we collect some definitions and preliminary propositions that will be used to prove Riemann-Roch theorem for curves.

Let K be an algebraically closed field over which the objects referred to in this paper will be defined. The *affine n-space over K* is the set of n-tuples

$$\mathbb{A}^n = \{(x_1, ..., x_n) | x_i \in K\}.$$

Let $K[X] = K[x_1, ..., x_n]$ denote a polynomial ring in n variables, and let $I \subseteq K[X]$ be an ideal in this polynomial ring. An *affine algebraic set* is a subset of $\mathbb{A}^n$ of the form

$$V(I) = \{P \in \mathbb{A}^n | f(P) = 0 \text{ for all } f \in I\}$$

for some ideal $I \subseteq K[X]$. If V is an algebraic set, the *ideal associated to* V is given by

$$I(V) = \{f \in K[X] | f(P) = 0 \text{ for all } P \in V\} \subseteq K[X]\}.$$

An affine algebraic set V is called an *(affine) variety* if $I(V)$ is a prime ideal in $K[X]$. For an affine variety V, the *coordinate ring of* V is defined by

$$K[V] = \frac{K[X]}{I(V)}$$

The *function field of* V is defined as the field of fractions of $K[V]$. We define the *dimension* of an algebraic variety V to be the Krull dimension of its coordinate ring $K[V]$. See [AM69, Chapter 1] for the definition of Krull dimension of a ring.

Let $f_1, ...f_m \in K[X]$ be a set of generators for an affine variety V, and let $P \in V$ be a point. We say that V is *smooth (nonsingular) at* P if the $m \times n$ matrix

$$\left(\frac{\partial f_i}{\partial X_j}(P)\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

has rank $n - \dim(V)$. We say that V is *smooth (nonsingular)* if V is smooth at every point.

For a variety V and a point $P \in V$, the *maximal ideal* of K[V] at P is

$$M_P = \{f \in K[V] | f(P) = 0\}.$$

Define the *local ring of* V *at* P, or $K[V]_P$, to be the localization of $K[V]$ at M_P:

$$K[V]_P = \{f \in K(V) | f = g/h \text{ for some } g, h \in K[V] \text{ and } h(P) \neq 0\}$$

Proposition 1. *Let C be a curve and $P \in C$ a smooth point. Then $K[C]_P$ is a discrete valuation ring.*

Proof. See [Sil86, Chapter II.1, Proposition 1.1]. □

Now, we will define similar terminology for projective n-space. *Projective n-space*, denoted $\mathbb{P}^n$, is the set of $n+1$-tuples

$$\{(x_0, ..., x_n) \in A^{n+1} | \text{ not all } x_i = 0\}$$

modulo the equivalence relation $(x_0, ..., x_n) \sim [y_0, ..., y_n]$ if there exists a constant λ in K such that $x_i = \lambda y_i \forall i$. An equivalence class of projective coordinates is denoted by $[x_0 : ... : x_n]$.

A polynomial $f \in K[x_1, ..., x_n]$ is *homogeneous of degree* d if

$$f(\lambda X_0, ..., \lambda X_n) = \lambda^d f(X_0, ..., X_n)$$

for all $\lambda \in K$. Similar to the affine case, we can define a *projective algebraic set* associated with a homogeneous ideal $I \subseteq K[x_0, ..., x_n]$ to be a subset of $\mathbb{P}^n$ of the form

$$V(I) = \{P \in \mathbb{P}^n | f(P) = 0 \text{ for all homogeneous } f \in I\}.$$

For a projective algebraic set V, the *homogeneous ideal of* V is

$$I(V) = \{f \in K[x_0, ..., x_n] | f \text{ is homogeneous and } f(P) = 0 \forall P \in V\}.$$

Furthermore, $K[V] = \frac{K[x_0,...,x_n]}{I(V)}$ and $K(V)$ is the same as the affine case with one additional property: if $f/g \in K(V)$, then f, g are homogeneous of the same degree.

We can define what is commonly known as a *Zariski topology* on $\mathbb{A}^n$ and $\mathbb{P}^n$ respectively by letting the closed sets be precisely the algebraic sets, and its complements in the respective $\mathbb{A}^n$ or $\mathbb{P}^n$ be the open sets. It is straightforward to check that the Zariski topology defined as before is indeed a topology. We call an algebraic variety *irreducible* if it cannot be written as the union of two proper closed subvarieties.
Clearly, $\mathbb{P}^n$ contains many copies of $\mathbb{A}^n$. For instance, for each $x_i \neq 0$, we let $H_i \subseteq \mathbb{P}^n$ denote the hyperplane given by $H_i = \{[x_0, ..., x_n] \in \mathbb{P}^n | x_i = 0\}$. From this, we can define an open set in the Zariski topology $U_i = \mathbb{P}^n \setminus H_i$. There is a natural bijection $\phi_i : U_i \longrightarrow \mathbb{A}^n$ such that $\phi_i(x_0, ..., x_n) = (\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i})$. An projective algebraic set V is called an *(projective) variety* if $I(V)$ is a prime ideal in $K[x_0, ..., x_n]$.

For a projective algebraic variety V with homogeneous ideal $I(V) \subseteq K[x_0, ..., x_n]$, we define an affine variety $V \cap \mathbb{A}^n = \phi_i^{-1}(V \cap U_i)$ so that $I(V \cap \mathbb{A}^n) \in K[x_0, ..., x_n]$ is generated by $\{f(x_0, ..., x_{i-1}, 1, x_{i+1}, ..., x_n) | f(x_0, ..., x_n) \in I(V)\}$. Observe that $\{U_0, ..., U_n\}$ cover all $\mathbb{P}^n$: we say that $\{U_0, ..., U_n\}$ is an *affine chart* for varieties on $\mathbb{P}^n$. The process of changing a variety from progective coordinates to affine coordinates is called *dehomogenization*, and the reverse process of going from affine coordinates to projective coordinates is called *homogenization*.

For a projective variety V, we define the *dimension* of V to be the dimension of $V \cap \mathbb{A}^n$. Similarly, the *function field* $K(V)$ is associated to the function field $K(V \cap \mathbb{A}^n)$. V is *nonsingular* if its respective $V \cap \mathbb{A}^n$ is nonsingular.

An irreducible projective variety of dimension one is a curve.

Let $V_1, V_2 \subseteq \mathbb{P}^n$. A *rational map between projective varieties* V_1 *and* V_2 is a map $\phi : V_1 \longrightarrow V_2$ that sends $P \in V_1$ to $\phi(P) = [f_0(P), ..., f_n(P)] \in V_2$ for each P for which all f_i are defined polynomials.

We can define a *normalized valuation on* $K[C]_P$ by

$$\text{ord}_P : K[C]_P \to \{0, 1, 2, ...\} \cup \{\infty\}$$

such that $\text{ord}_P(f) = \sup\{d \in \mathbb{Z} | f \in M_P^d\}$. We can extend this valuation to $K(C)$ by letting $\text{ord}_P(f/g) = \text{ord}_P(f) - \text{ord}_P(g)$ for $f \in K[C]$ and $g \in K[C]_P$. $\text{ord}_P(f)$ is called the *order of* f *at* P. A *uniformizer* for C at P is any function $f \in K(C)$ such that $\text{ord}_P(f) = 1$.

Suppose that $\text{ord}_P(f) = n$. If $n \geq 0$, then f is *regular* at P. If $n < 0$, then f has a *pole of order* n at P; we write $f(P) = \infty$. If $n > 0$, then f has a *zero* at P.

Proposition 2. *Let C be a smooth curve, let $f \in K(C)$ and $f \neq 0$. Then f is a pole or zero at only finitely many points.*

Proof. See [Har77, Chapter I, 6.5]. □

Proposition 3. *Let f be as before. If f has no poles, then $f \in K$.*

Proof. Consider $1/f$. Since f has no poles, then $1/f$ has no zeros, therefore it is constant. If $1/f$ is constant, then f must be constant. □

The *divisor group of C, Div(C)*, is the free abelian group generated by the points of C. A divisor $D \in \mathrm{Div}(C)$ takes the form

$$\sum_{P \in C} n_P(P)$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in C$. The *degree* of D is

$$\sum_{P \in C} n_P$$

. If C is a smooth curve and $f \in K(C)$, we associate to f the divisor

$$\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f)(P)$$

Since each ord_P is a valuation, the map

$$\mathrm{div} : K(C) \to \mathrm{Div}(C)$$

is a homomorphism of abelian groups.

We say that a divisor D is *principal* if there exists an $f \in K(C)$ for which $D = \mathrm{div}(f)$. Two divisors D_1, D_2 are *linearly equivalent*, denoted $D_1 \tilde{D}_2$ if $D_1 - D_2$ is principal. The *Picard group of C, Pic*(C) is the quotient of $\mathrm{Div}(C)$ by its subgroup of principal divisors.

Proposition 4. *Let C be a smooth curve, and let $f \in K(C)$. Then $div(f) = 0$ if and only if $f \in K$, and* $\deg(div(f)) = 0$.

Proof. If $\mathrm{div}(f) = 0$, then f has no poles. By Proposition 2, f is constant. The converse is clear. For the proof of the second part, see [Har77, Chapter II, 6.10]. □

For an explanation of differential forms on a smooth curve C, see [Sil86, Chapter II. §4].

Here, we say that a *holomorphic 1-form* on a curve C is a differential form ω of degree 1 on C that can be written locally as $\omega = f dx$ such that f is regular with respect to the local coordinate x. We define a *meromorphic 1-form* on a curve C to be a differential form of degree 1 that is regular on $X \setminus S$ for some discrete, closed, possibly empty subset $S \subset X$, such that any point $a \in S$ has a neighborhood U on which the restriction of ω can be written as $f\omega'$, where f is a rational function on U with a pole at a and w' is a holomorphic 1-form on U. We denote such $f\omega'$ as $Res\omega$.

If ω is a meromorphic 1-form on C that can be locally represented as some $f_a dx$ in the neighborhood of each point $a \in C$, we define $\text{ord}_P(\omega) = \text{ord}_P(f_P)$. The divisor

$$\text{div}(\omega) = \sum_{P \in C} \text{ord}_P(\omega)(P)$$

is called the *canonical divisor* of the meromorphic 1-form ω on the curve C.

We call a divisor $D = \sum n_P(P)$ *effective*, denoted $D \geq 0$ if $n_P \geq 0 \forall P \in C$. For two divisors D_1, D_2, we say $D_1 \geq D_2$ to indicate that $D_1 - D_2$ is effective.
We define a vector space of functions associated with a divisor $D \in \text{Div}(C)$ to be:

$$\mathcal{L}(D) = \{f \in K(C) | \text{div}(f) \geq -D\} \cup \{0\}.$$

This set is a finite dimensional K-vector space, and we denote

$$\ell(D) = \dim_K \mathcal{L}(D).$$

Proposition 5. *Let $D \in Div(C)$. If $\deg D < 0$, then $\mathcal{L} = 0$ and $\ell(D) = 0$. Furthermore, if $D' \in Div(C)$ is linearly equivalent to D, then $\mathcal{L}(D) \cong \mathcal{L}(D')$ and $\ell(D) = \ell(D')$*

Proof. Let $f \in \mathcal{L}(D)$ with $f \neq 0$. Then, $\deg \text{div}(f) = 0 \geq \deg(-D) = -\deg(D)$, so $\deg(D) \geq 0$.
Next, if $D = D' + \text{div}(g)$ for some $g \in K(C)$, then the map $\mathcal{L}(D) \to \mathcal{L}(D')$ such that $f \mapsto fg$ is an isomorphism of vector spaces. □

We define the *principal part* of a function

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

at $z = a$ to be the portion of the Laurent series consisting of terms with negative degree. In other words,

$$\sum_{k=-\infty}^{-1} a_k (z-a)^k$$

is the principal part of the given f at a.

3. Proof of the theorem

In this section, we give a proof of the Riemann-Roch theorem for curves.

Proof of Theorem 1. First we consider the case of effective divisors. Let C be a curve of genus g. Let K be the class of canonical divisors on C, and let

$$D = \sum_{m=1}^{n} m_i P_i$$

be a (positive) effective divisor. Let

$$V = \left\{ (f_1, ..., f_n) \mid f_i = \frac{c_{m_i}}{z^{m_i}} + ... + \frac{c_{-1}}{z} \right\}$$

namely, the set of all tuples of functions on $K(C)$ which contain poles of highest degree at most m_i for each i. Clearly, V is a linear space over K of dimension $\deg(D)$.

Define a map $\phi : \mathcal{L}(D) \to V$ that sends $f \in \mathcal{L}(D)$ to the tuple of principal parts of f at the points P_i. Consider $\ker\phi$, which is a subset of functions in $\mathcal{L}(D)$ that are sent to zero. Since $D \geq 0$, a function f with $\text{div}(f) \geq -D$ that is in this kernel have no principal parts at P_i and therefore no other poles; f is regular everywhere and therefore has no poles; by Proposition 3, f is constant. Therefore, $\dim\ker\phi = 1$, consisting of only the constant functions.

Let $\text{Im}(\phi) = W$. We have

$$\ell(D) = \dim(\ker\phi) + \dim(\text{Im}(\phi)) = 1 + \dim(W)$$

Consider $\dim W$. W is a set of $\{f_i\}$ principal parts such that there exists $f \in \mathcal{L}(D)$ with set of tails equal to $(f_1, \ldots, f_n)$. According to (X), such f exists if and only if for all holomorphic 1-forms ω on C,

$$\sum_{i=1}^{n} Res_{a_i} f_i \omega = 0$$

For each holomorphic differential 1-form ω, consider the linear map $\lambda_\omega : V \to K$ such that $\{f_1, ..., f_n\}$ is sent to

$$\sum_{i=1}^{n} Res_{a_i} f_i \omega = 0$$

From this definition, $W = \cap \ker\lambda_\omega$ is the intersection of the kernels of λ_ω over all ω. It follows that if $\{\lambda_\omega\}$ is the linear space generated by all λ_ω, then

$$\dim W = \dim \cap \ker(\lambda_\omega) = \dim(V) - \dim(\{\lambda_\omega\})$$

.
We know that $\dim(V) = \deg(D)$.
So the expression becomes

$$\dim(W) = \deg(D) - \dim(\{\lambda_\omega\})$$

and

$$\ell(D) = 1 + \dim(W) = 1 + \deg(D) - dim(\{\lambda_\omega\})$$

.

Since we know that the number of linearly independent holomorphic 1-forms in the space of differentials on C is g, it follows that $\dim(\{\lambda_\omega\}) \leq g$. We need only consider the differential 1-forms that turn all principal parts to 0, as these will correspond precisely to the maps λ_ω that do not contribute to the space generated by $\{\lambda_\omega\}$.

The principal part $\frac{c_{m_i}}{z^{m_i}} + ...$ at a_i turns to zero when multiplied by a differential ω such that $\text{ord}_{a_i}(\omega) \geq m_i$. This will happen only when $\text{div}(\omega) \geq D$, in other words, when $\omega \in \mathcal{L}(K_C - D)$. Therefore, $\lambda_\omega = 0$ if and only if $\omega \in \mathcal{L}(K_C - D)$, so

$$\dim(\{\lambda_\omega\}) = g - \ell(K_C - D)$$

From this, it follows that

$$\ell(D) = 1 + deg(D) - g + \ell(K_C - D)$$

as desired.

Next, we look at the case of the general divisor and show that

$$\ell(D) - \ell(K_C - D) \geq 1 + \deg(D) - g$$

Let $x \in C$ be a point on the curve; it follows immediately that

$$\deg(D - a) = \deg(D) - 1$$

as it is just a subtraction of integers. If the inequality holds for any divisor D, then

$$\ell(D-a)-\ell(K_C-(D-a)) \geq 1+\deg(D-a)-g = \deg(D)+1-g-1 = \ell(D)-\ell(K_C-D)-1$$

implying the inequality is true for $D - a$ as well.

From this, our argument is inductive in nature, where we subtract from a divisor point-by-point. Combined with the equality for positive effective divisors proved in the previous part, we only need to show that

$$\ell(D - a) - \ell(K_C - (D - a)) \geq (\ell(D) - \ell(K_C - D)) - 1$$

Clearly,

$$\ell(D) \geq \ell(D - a) \geq \ell(D) - 1$$

and

$$\ell(K_C - D) + 1 \geq \ell(K_C - (D - a)) \geq \ell(K_C - D)$$

Therefore, the worst case scenario occurs when

$$\ell(D - a) = \ell(D) - 1$$

and

$$\ell(K_C - D) + 1 = \ell(K_C - (D - a))$$

Only in this case,

$$\ell(D - a) - \ell(K_C - (D - a)) = (\ell(D) - \ell(K_C - (D - a))) - 2$$

Because when we are not in the worst case scenario, either

$$\ell(D - a)\ell(K_C - (D - a)) = (\ell(D) - \ell(K_C - (D - a))) - 1$$

or

$$\ell(D - a)\ell(K_C - (D - a)) = (\ell(D) - \ell(K_C - (D - a)))$$

To show that the worst case scenario is impossible, take $f \in \mathcal{L} \setminus \mathcal{L}(D - a)$ and $\omega \in \mathcal{L}(K_C - (D - a)) \setminus \mathcal{L}(K_C - D)$. This implies that $\text{div}(f) \geq -D$, $\text{div}(f) < a - D$, div $(\omega) \geq D - a$, and $\text{div}(\omega) < D$. By the assumption that $\ell(D - a) = \ell(D) - 1$ and $\ell(K_C - D) + 1 = \ell(K_C - (D - a))$, such f and ω must exist. Suppose that $D = n \cdot a + \ldots$. This implies that $-n + 1 > \text{ ord}_a(f) \geq -n$, therefore $\text{ord}_a(f) = -n$. Similarly, $n > \text{ ord}_a(\omega) \geq n - 1$, so $\text{ord}_a(\omega) = n - 1$. So, $\text{ord}_a(f \cdot \omega) = -n + n - 1 = -1$.

For all $b \neq a$, $\text{ord}_b(f \cdot \omega) \geq 0$ because $\text{div}(f) \geq -D$ and $\text{div}(f \cdot \omega) \geq -a$, implying that the poles may occur only at a.

However, now we have both

$$\sum_{i=1}^{n} Res_{a_i} f \cdot \omega = 0$$

and $Res_a(f \cdot \omega) = c_{-1} \neq 0$ for some $c_{-1} \neq 0$, as we have a pole of exactly order 1 at a, which is the only point where $\text{div}(f \cdot \omega)$ has non-zero residue. However, this leads to the contradiction, and this scenario can never occur.

Therefore, $\ell(D-a) - \ell(K_C - (D-a)) \geq (\ell(D) - \ell(K_C - D)) - 1$. This implies that for all divisors,

$$\ell(D) - \ell(K_C - D) \geq 1 + \deg(D) - g$$

Finally, we wish to arrive at the desired equality in the previously given statement for the arbitrary divisor D. Substitute $K_C - D$ for D in the inequality proven in Part II. This gives us the following inequalities:

$$\ell(K_C - D) - \ell(K_C - (K_C - D) \geq \deg(K_C - D) + 1 - g$$
$$\ell(K_C - D) - \ell(D) \geq \deg(K_C - D) + 1 - g$$
$$\ell(K_C - D) - \ell(D) \geq \deg(K_C) - \deg(D) + 1 - g$$

Since $\deg(K_C) = \deg(\mathrm{div}(\omega)) = 2 - 2g$, we have

$$\ell(K_C - D) - \ell(D) \geq 2g - 2 - \deg(D) + 1 - g = g - 1 - \deg(D)$$
$$\ell(D) - \ell(K_C - D) \leq \deg(D) + 1 - g$$
$$\ell(D) - \ell(K_C - D) = deg(D) + 1 - g$$

Combining this with the inequality proved in the second part, $\ell(D) - \ell(K_C - D) \geq 1 + \deg(D) - g$, we conclude the Riemann-Roch theorem. □

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