Modular Forms and Elliptic Curves

Sriram Gopalakrishnan

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Abstract

We define what it means for a function to be a modular form, prove fundamental theorems about modular forms, and explore the relationship between lattice functions and modular forms. We conclude by giving a small glimpse of the relationship between modular forms and elliptic curves.

1 Introduction

In this paper, we focus on the fundamentals of modular forms. We first give a definition of a modular form through a careful series of theorems and proofs. We follow with a discussion of lattices and their relations to modular forms. We give a common example of a lattice function that is also a modular form, namely Eisenstein series and we conclude by providing some motivations for the study of modular forms by exploring their relationship with elliptic curves. This paper follows Chapter 7 of J.P. Serre's text *A Course in Arithmetic*, while filling in details that Serre omits. The final proof follows that of Washington in his book *Elliptic Curves, Number Theory, and Cryptography*.

2 Preliminary

2.1 The Modular Group

Recall the definition of the special linear group of 2×2 matrices over the integers:

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

We call $G = SL_2(\mathbb{Z})/\pm 1$ where 1 is the identity element (the identity matrix) in $SL_2(\mathbb{Z})$ the modular group. The modular group forms the starting point for our discussion of modular forms, so it is beneficial to state and prove some facts about it. We define the complex upper half plane $\mathcal{H}$ as follows:

$$\mathcal{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$$

What follows is a discussion and development of the G-action on $\mathcal{H}$

Let $g \in G$ be a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, for every $g \in G$, $z \in \mathcal{H}$, the action of g on z is given by

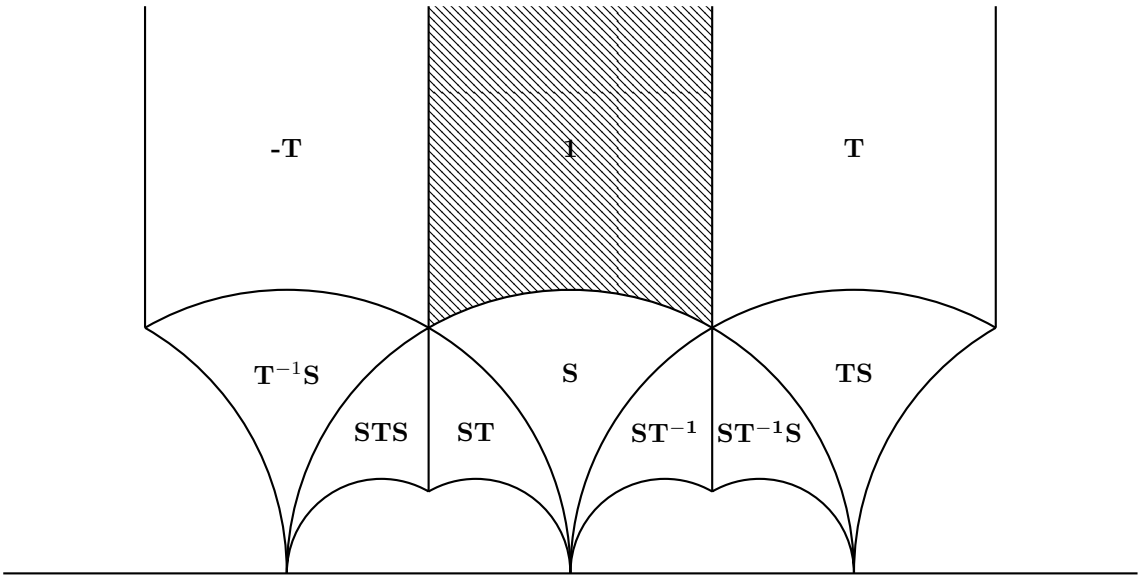
$$gz = \frac{az + b}{cz + d}$$

Let $S, T \in G$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ The S and T actions on a complex variable $z \in \mathcal{H}$ are then

$$Sz = -\frac{1}{z} \qquad Tz = z + 1$$

Furthermore, $S^2 = 1$ and $(ST)^3 = 1$. Now, let $D := \left\{ z \in \mathcal{H} \;\middle|\; |z| \geq 1, |Re(z)| \leq \frac{1}{2} \right\}$. The following figure represents the transformations of D by the elements:

$$\{1, T, -T, T^{-1}S, STS, ST, S, ST^{-1}, ST^{-1}S, TS\} \in G$$



Theorem 2.1.1. *The modular group G is generated by the elements S and T.*

Proof. We will give an explicit expression of an arbitrary element $g \in G$ in terms of S and T. For some matrix in G, we have the following multiplications on the left by the elements S and T.

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix} \tag{1}$$

Now let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary element in G. Suppose $c \neq 0$. If $|a| > |c|$, write $a = cq + r$ (division algorithm), where $0 \leq r < |c|$. By (1), the upper left entry of the matrix $T^{-q}g$ is $a - qc = r < |c|$. We can apply S to $T^{-q}g$ to swap the rows of $T^{-q}g$. Once more, if our new matrix has a nonzero entry in the bottom left spot, we can perform the division algorithm again to obtain a new power of T by which we can multiply $T^{-q}S$ by to again reduce the element in the bottom left entry. We can continue this process until we obtain a matrix with lower left entry 0. Since this matrix must still be in G (as it is a product of matrices in G), it must have integer entries and determinant 1. Thus, it is of the form

$$\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}$$

where $m \in \mathbb{Z}$ and the diagonal entries have the same sign. This matrix is either T^m or $-T^{-m}$, so there exists $h \in G$ that is the multiple of powers of T and copies of S such that $hg = \pm T^n$ for some $n \in \mathbb{Z}$. As $T^n \in G$ and $S^2 = I$, we can write $g = \pm h^{-1}T^n$ where h^{-1} is generated by S and T and we are done. $\square$

Theorem 2.1.2. *Let D, G, and $\mathcal{H}$ be as above.*

(1) For every $z \in \mathcal{H}$, there exists $g \in G$ such that $gz \in D$.

(2) Suppose $z, z' \in \mathcal{H}$ are congruent modulo the G-action. That is, suppose z and z' have the same orbit under the G-action. Then,

(i) $Re(z) = \pm\frac{1}{2}$ and $z = z' \pm 1$ or

(ii) $|z| = 1$ and $z' = -\frac{1}{z}$

(3) Fix $z \in D$ and let $G_z = \{g \in G \mid gz = z\}$ be the stabilizer of z under the G-action. $G_z = \{1\}$ except if:

(i) $z = i$, in which case G_z is generated by S and is a group of order 2;

(ii) $z = e^{\frac{2\pi i}{3}}$, in which case G_z is generated by ST and is a group of order 3;

(iii) $z = e^{\frac{\pi i}{3}}$, in which case G_z is generated by TS and is a group of order 3.

Proof. Let $G' \leq G$ be the subgroup of G generated by S and T. Let $g \in G'$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $z \in \mathcal{H}$, $z = r + si$. Then,

$$\begin{aligned} Im(gz) &= Im\left(\frac{az+b}{cz+d}\right) = Im\left(\frac{ar+asi+b}{cr+csi+d}\right) = Im\left(\frac{(ar+b)+asi}{(cr+d)+csi}\right) \\ &= Im\left(\frac{(ar+b)(cr+d)-csi(ar+b)+asi(cr+d)}{|cz+d|^2}\right) = Im\left(\frac{(da-bc)si}{|cz+d|^2}\right) \\ &= \frac{Im(z)}{|cz+d|^2} \end{aligned}$$

Fix $M \in \mathbb{Z}$. Then, $|\{(c,d) \in \mathbb{Z}^2 \mid |cz+d| < M\}| < \infty$. Thus, there exists a pair (c,d) such that $|cz+d|$ is minimal and consequently $\frac{Im(z)}{|cz+d|^2}$ is maximal. In other words, we can find $h \in G'$ such that $Im(hz) = \frac{Im(z)}{|cz+d|^2}$ is maximized. We now choose an integer n such that $|Re(T^n hz)| \leq \frac{1}{2}$. Let $z' = T^n hz$. Then,

$$Im\left(\frac{-1}{z'}\right) = \frac{Im\,(z')}{|z'|^2}$$

so if $|z'| < 1$, $Im\left(\frac{-1}{z'}\right) > Im\,(z')$. As this is impossible, $|z'| \geq 1$. Therefore, $g' = T^n hz$ is in the fundamental domain D. Let $z \in D$ and choose $g \in G$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $gz \in D$. Suppose $Im(gz) \leq Im(z)$. Then, replace the pair (z, g) with (gz, g^{-1}) to obtain $Im(g^{-1}gz) = Im(z) \leq Im(gz)$. We can thus assume without loss of generality, that $Im(gz) \geq Im(z)$, so $|cz+d| \leq 1$. This is only possible if $|c| < 2$, so we consider the cases $c = -1, 0, 1$.
Suppose $c = 0$. Since g is an element of the modular group, its determinant must be 1. Therefore,

$d = \pm 1$. Therefore, $gz = z \pm b$. Since $z, gz \in D$, $|Re(z)| \leq \frac{1}{2}$ and $|Re(gz)| \leq \frac{1}{2}$. Hence, we have

$$b = \begin{cases} 0 & \text{if } g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, |Re(z)| = |Re(gz)| < \frac{1}{2} \\ -1 & \text{if } Re(z) = \frac{1}{2}, Re(gz) = -\frac{1}{2} \\ 1 & \text{if } Re(z) = -\frac{1}{2}, Re(gz) = \frac{1}{2} \end{cases}$$

If $c = 1$, $|z + d| \leq 1$, so $d = 0$, unless

$$z = \begin{cases} e^{\frac{2\pi i}{3}} \implies d = 0 \text{ or } 1 \\ e^{\frac{\pi i}{3}} \implies d = 0 \text{ or } -1 \end{cases}$$

When $d = 0$, $|z| = 1$, and $b = -1$. Therefore, $gz = a - \frac{1}{z}$. Thus, $a = 0$ except when $Re(z) = \pm\frac{1}{2}$ (equivalently when $z = e^{\frac{2\pi i}{3}}$ or $z = e^{\frac{\pi i}{3}}$), in which case $a = 0, -1$ or $a = 0, 1$. When $z = e^{\frac{2\pi i}{3}}$ and $d = 1$, $a - b = 1$ and $\frac{a-1}{1+e^{\frac{2\pi i}{3}}} = a + e^{\frac{2\pi i}{3}}$, implying that $a = 0, 1$. Similarly, we find that when $z = e^{\frac{\pi i}{3}}$ and $d = -1$, a must be 0 or -1. Suppose $c = -1$. We can change the signs of all a, b, c, d in g, which does not change the g-action on an element in $\mathcal{H}$, and follow the same argument as the case when $c = 1$.

□

2.2 Lattices

We begin with the definition of a lattice. There are many equivalent definitions, but we will use the following one for the sake of this paper.

Definition 2.2.1. *Let V be a finite dimensional real vector space. A lattice Γ of V is a discrete subgroup of V (where V is viewed as a group) that generates V.*

Consider the vector space $\mathbb{C}$ over the field $\mathbb{R}$. Let $\mathcal{R}$ be the set of lattices of $\mathbb{C}$. Let

$$M := \left\{ (\omega_1, \omega_2) \in \mathbb{C}^2 \mid Im\left(\frac{\omega_1}{\omega_2}\right) > 0 \right\}$$

The condition that $Im\left(\frac{\omega_1}{\omega_2}\right) > 0$ is equivalent to saying that ω_1 and ω_2 cannot both be real. We associate the lattice $\Gamma(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with the lattice basis $\{\omega_1, \omega_2\}$. Suppose

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad (\omega_1, \omega_2) \in M$$

Then, the pair $\{\omega_1', \omega_2'\}$ where $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ forms a different lattice basis for $\Gamma(\omega_1, \omega_2)$. Indeed, let $z = \frac{\omega_1}{\omega_2}$ and $z' = \frac{\omega_1'}{\omega_2'}$. Then,

$$z' = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{a\frac{\omega_1}{\omega_2} + b}{c\frac{\omega_1}{\omega_2} + d} = \frac{az + b}{cz + d} = gz$$

Theorem 2.2.1. *Two elements of M define the same lattice if and only if they are congruent modulo the $SL_2(\mathbb{Z})$ action.*

Proof. We have shown above that if (ω_1, ω_2) and (ω_1', ω_2') are congruent modulo the $SL_2(\mathbb{Z})$ action, then they define the same lattice. We will now show the converse. Suppose (ω_1, ω_2) and (ω_1', ω_2') are both in M and both define the same lattice. Then, since (ω_1, ω_2) lies in the lattice generated by (ω_1', ω_2'), we can write

$$\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix} = \begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{bmatrix}\omega_1' \\ \omega_2'\end{bmatrix} = P\begin{bmatrix}\omega_1' \\ \omega_2'\end{bmatrix}$$

We can represent (ω_1', ω_2') as a linear combination of (ω_1, ω_2) likewise. If we invert P, we get

$$P^{-1}\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix} = \begin{bmatrix}\omega_1' \\ \omega_2'\end{bmatrix}$$

Thus, $det(P) = det(P^{-1} = det(P)^{-1}$, where P and P^{-1} both have integer entries. Hence, $det(P) = \pm 1$. Suppose $det(P) = -1$. Then, $Im\left(\frac{\omega_1}{\omega_2}\right) = -Im\left(\frac{\omega_1'}{\omega_2'}\right)$. But since both (ω_1, ω_2) and (ω_1', ω_2') are in M, the signs must be the same. Thus, $det(P) = 1$, and so $P \in SL_2(\mathbb{Z})$. □

Due to Theorem 2.2.1, we can write the definition of $\mathcal{R}$ as $M/SL_2(\mathbb{Z})$. Now, suppose we start with some $(\omega_1, \omega_2) \in M$. As usual, we call the lattice generated by (ω_1, ω_2) $\Gamma(\omega_1, \omega_2) \in \mathcal{R}$. Scale the pair (ω_1, ω_2) by $\lambda \in \mathbb{C}^\times$ to arrive at $(\lambda\omega_1, \lambda\omega_2)$. Then, the lattice generated by $(\lambda\omega_1, \lambda\omega_2)$ is $\lambda\Gamma$ and is equivalent to the action of an element in $SL_2(\mathbb{Z})$ on Γ. In other words, the action of scaling on M commutes with the action of $SL_2(\mathbb{Z})$ on $\mathcal{R}$. Thus, the map

$$(\omega_1, \omega_2) \mapsto z = \frac{\omega_1}{\omega_2} \tag{2}$$

gives a transformation of the action of $SL_2(\mathbb{Z})$ on M to the G action on H. Furthermore, the map given by (2) defines a bijection between $\mathcal{R}/\mathbb{C}^\times$ and H/G. Recall that we can associate to each Γ in $\mathbb{C}$ an elliptic curve $E_\Gamma = \mathbb{C}/\Gamma$. If $\Gamma' = \lambda\Gamma$, then the elliptic curves E_Γ and $E_{\Gamma'}$ are isomorphic. This gives us yet another equivalent description of H/G - namely that it is the set of isomorphism classes of elliptic curves. Particularly, two elliptic curves are isomorphic if the lattices defining them are in the same equivalence class in $\mathcal{R}/\mathbb{C}^\times$.

3 Modular Forms and Lattice Functions

3.1 Modular Forms

We begin with a short description of what a meromorphic complex function is.

Definition 3.1.1. *A complex-valued function $f(z)$ is meromorphic on an open set D of the complex plane if it is holomorphic (resp. complex differentiable, analytic) on all of D except for a finite set of isolated points, called poles.*

A meromorphic function is the ratio of two holomorphic functions. That is, if f and g are holomorphic functions, then $\frac{f(z)}{g(z)}$ is meromorphic and has poles at the zeroes of $g(z)$. With this brief recollection of the idea of a meromorphic function, we are ready to give our first major definition pertaining to modular forms.

Definition 3.1.2. *A weakly modular function of weight $2k$ (resp. k, $-2k$) where $k \in \mathbb{Z}$ is a function that is meromorphic on $\mathcal{H}$ and satisfies*

$$f(z) = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right) \text{ for all } \begin{pmatrix}a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$$

Theorem 3.1.1. *Suppose f is a meromorphic function on $\mathcal{H}$. Then f is a weakly modular function of weight $2k$ if and only if it satisfies both*

$$f(z+1) = f(z) \tag{3}$$
$$f\left(\frac{-1}{z}\right) = z^{2k} f(z) \tag{4}$$

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and consider the derivative of gz, with respect to z

$$\begin{aligned} \frac{d(gz)}{dz} = \frac{d\left(\frac{az+b}{cz+d}\right)}{dz} &= \frac{a(cz+d) - c(az+b)}{(cz+d)^2} \\ &= \frac{acz + ad - acz - cb}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \\ &= \frac{1}{(cz+d)^2} \end{aligned}$$

Thus, we can write

$$\frac{f(gz)}{f(z)} = \left(\frac{d(gz)}{dz}\right)^{-k} \tag{5}$$

By Theorem 2.1.1, the modular group G is generated by the elements S and T. Thus, if f is a meromorphic function on $\mathcal{H}$, and (5) is satisfied for S and T, then f is a weakly modular function of weight $2k$. For S, we can write (5) as $f\left(\frac{-1}{z}\right) = z^{2k} f(z)$, so (4) is satisfied. Likewise for T we have $f(z+1) = f(z)$, so (3) is satisfied. The same argument in reverse works to show the converse. □

Suppose (3) is satisfied. Then, f is a periodic function of period 1, so we can represent f with a Fourier series. That is, f can be represented as a function of $q = e^{2\pi i z}$. Let $\tilde{f}$ be this representation of f. f is meromorphic on $\mathcal{H}$, so $\tilde{f}$ is meromorphic on the (punctured) unit disk $|q| < 1$ with the origin removed. Under the conformal mapping that takes f to $\tilde{f}$ (resp, $z \mapsto e^{2\pi i z}$), the point $i\infty$ gets sent to $e^{2\pi i(i\infty)} = e^{-2\pi\infty} = 0$. Thus, if $\tilde{f}$ extends to a meromorphic (resp. holomorphic) function at the origin, we say that f is meromorphic (resp. holomorphic) at infinity. In other words, if $\tilde{f}$ extends to a meromorphic (resp. holomorphic) function at the origin, then $\tilde{f}$ has a Laurent expansion in some neighborhood of the origin given by

$$\tilde{f}(q) = \sum_{-\infty}^{\infty} a_n q^n$$

where $a_n = 0$ when $n < 0$. Finally, we come to the definition of modular functions and modular forms.

Definition 3.1.3. *If f is a weakly modular function and extends to a meromorphic function at infinity, we call f a modular function. In this case, we set $f(\infty) = \tilde{f}(0)$.*

Definition 3.1.4. *Suppose f is a complex-valued function. Then, we call f a modular form if*

(i) $f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$,

(ii) f is holomorphic on $\mathcal{H}$, and

(iii) f is holomorphic at ∞.

In this case, we once again set $f(\infty) = \tilde{f}(0)$. If $f(\infty) = \tilde{f}(0) = 0$ (that is, if f assumes the value of 0 at ∞, or equivalently if the constant term in the Laurent expansion of $\tilde{f}(q)$ is 0), then we call f a modular cusp form.

3.2 Lattice Functions

Once again, we begin with a definition.

Definition 3.2.1. *Suppose F is a function on $\mathcal{R}$ (the set of lattices of $\mathbb{C}^\times$) and $k \in \mathbb{Z}$. F is a modular lattice function of weight $2k$ if*

$$F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma) \tag{6}$$

for all lattices $\Gamma \in \mathcal{R}$ and all $\lambda \in \mathbb{C}^\times$.

Now suppose F is a lattice function of weight $2k$ and let $(\omega_1, \omega_2) \in M$. $F(\omega_1, \omega_2)$ is the value of F on the lattice $\Gamma(\omega_1, \omega_2)$. Then, (6) gives us

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} F(\omega_1, \omega_2) \tag{7}$$

Let $\lambda = \omega_2$. By the map given in the preliminary lattice section by (2) $z = \frac{\omega_1}{\omega_2}$, there exists a function f on $\mathcal{H}$ such that

$$F(\omega_1, \omega_2) = \omega_2^{-2k} f(z) \tag{8}$$

If we say that F is invariant under $SL_2(\mathbb{Z})$,

$$f(z) = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{9}$$

Likewise, if a function f satisfies (9), then by (8), we can identify f with a function F on $\mathcal{R}$ of weight $2k$. This gives us our main result, namely that modular lattice functions of weight $2k$ can be identified with modular functions of weight $2k$ and vice versa.

3.3 Eisenstein Series

We begin with the definition of Eisenstein series.

Definition 3.3.1 (Eisenstein Series)**.** *Let $k \in \mathbb{Z}$, $k > 1$ and let Γ be as usual, a lattice in $\mathbb{C}^\times$. Then,*

$$G_k(\Gamma) = \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^{2k}}$$

We call G_k the Eisenstein series of index k.

This series is absolutely convergent (see Serre). Furthermore,

$$G_k(\lambda\Gamma) = \sum_{0 \neq \gamma \in \Gamma} \frac{1}{(\lambda\gamma)^{2k}} = \lambda^{-2k} \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^{2k}} = \lambda^{-2k} G_k(\Gamma)$$

Therefore, we see that G_k is a lattice function of weight $2k$. We can thus view G_k as a function on M.

$$G_k(\omega_1, \omega_2) = \sum_{(0,0)\neq(m,n)} \frac{1}{(m\omega_1 + n\omega_2)^{2k}}$$

Now consider, as in the lattice preliminary section, the map $(\omega_1, \omega_2) \mapsto z = \frac{\omega_1}{\omega_2}$. We can write

$$\begin{aligned} G_k(\omega_1, \omega_2) &= \sum_{(0,0)\neq(m,n)} \frac{1}{\left(\frac{1}{\omega_2}\right)^{2k} (m\omega_1 + n\omega_2)^{2k}} \\ &= \sum_{(0,0)\neq(m,n)} \frac{1}{(mz+n)^{2k}} \\ &= G_k(z) \end{aligned} \tag{10}$$

Here, $G_k(z)$ differs from $G_k(\omega_1, \omega_2)$ by a factor of ω_2^{2k}.

Theorem 3.3.1. *Let $k \in \mathbb{Z}$, $k > 1$. Then, $G_k(z)$ is a modular form of weight $2k$ with $G_k(\infty) = 2\zeta(2k)$.*

Proof. G_k is convergent (see Serre). Thus, in conjunction with the agrument that G_k is a lattice fuction of weight $2k$, G_k is weakly modular of weight $2k$. Suppose now that $z \in D$, the fundamental domain of the G-action on $\mathcal{H}$. Then, we have

$$|mz + n|^2 = m^2 z\bar{z} + 2mnRe(z) + n^2 \geq m^2 - mn + n^2 = |m\rho - n|^2$$

where $\rho = e^{\frac{2\pi i}{3}}$. The series $\sum_{(0,0)\neq(m,n)} \frac{1}{|m\rho - n|^{2k}}$ converges in D (see Serre) and thus in gD for $g \in G$. The set $\{gD | g \in G\}$ covers $\mathcal{H}$, so G_k is holomorphic on all of $\mathcal{H}$. We now show that G_k has a limit as $z \to i\infty$, and thus G_k is "holomorphic at infinity." We can again suppose $Z \in \mathcal{H}$. The terms of G_k for which $m \neq 0$ give us $G_k(z) \to 0$. The terms of G_k for which $m = 0$ give us:

$$\lim_{z \to i\infty} G_k(z) = \sum_{n \neq 0} \frac{1}{n^{2k}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k)$$

□

Having developed an understanding of Eisenstein series as a modular form, we now describe a modular cusp form given by Eisenstein series of particular weights. Suppose we have the following:

$$g_2 = 60G_2, \qquad g_3 = 140G_3$$

where G_2 is the Eisenstein series of weight 4 and G_3 is the Eisenstein series of weight 6. We have

$$g_2(\infty) = 120\zeta(4), \qquad g_3(\infty) = 280\zeta(6)$$

by Theorem 3.3.1. The values of $\zeta(4)$ and $\zeta(6)$ are $\frac{\pi^4}{90}$ and $\frac{\pi^6}{945}$, respectively. Thus,

$$g_2(\infty) = \frac{4}{3}\pi^4, \qquad g_3(\infty) = \frac{8}{27}\pi^6$$

Now let

$$\Delta = g_2^3 - 27g_3^2$$

Then,

$$\Delta(\infty) = g_2(\infty)^3 - 27g_3(\infty)^2 = \frac{64}{27}\pi^{12} - \frac{64}{27}\pi^{12} = 0$$

Since g_2 and g_3 are modular forms, Δ is a modular form. Furthermore, it assumes the value 0 at infinity. Thus, it is a modular cusp form. Note that we do not know from this alone that the function Δ is not the 0 function. In the next section, we explain why Δ is not identically zero.

4 Modular Forms and Elliptic Curves

4.1 The Weierstrass $\wp$-function

In this section, we will ignore issues of convergence of the $\wp$ function.

Definition 4.1.1. *Let Γ be a lattice in $\mathbb{C}^\times$. Then, the Weierstrass $\wp$-function corresponding to Γ is*

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{0 \neq \gamma \in \Gamma} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right)$$

We denote the negative of the antiderivative of $\wp(z)$ as $\zeta_\wp(z)$. We have the explicit formulation of $\zeta_\wp(z)$

$$\zeta_\wp(z) = \frac{1}{z} + \sum_{\gamma \neq 0} \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right)$$

We can expand the summand to the following

$$\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} = -\frac{z^2}{\gamma^3} - \frac{z^3}{\gamma^4} - \frac{z^4}{\gamma^5} \cdots$$

Thus,

$$\zeta_\wp(z) = \frac{1}{z} - \sum_{k=2}^{\infty} G_k(z) z^{2k-1}$$

where $G_k(z)$ is the Eisenstein series of index k. The odd powers have disappeared here since $\wp(z)$ is an even function. Note that $\wp(z) = -\zeta'_\wp(z)$, so

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k-1) G_k(z) z^{2k-2} \tag{11}$$

In terms of lattices, we can rewrite (9) as

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k-1) G_k(\Gamma) z^{2k-2}$$

We now turn to the relationship between the Weierstrass $\wp$-function and elliptic curves. We by stating a theorem without proof that will be necessary to prove subsequent propositions.

Theorem 4.1.1. *Let f be a doubly-periodic function on the lattice Γ and let F be the fundamental parallelogram of Γ (the paralellogram who's left and bottom sides are (ω_1, ω_2), the basis for Γ). Then,*

1. *If f has no poles, then f is constant*

2. *If f is not constant, then $f : \mathbb{C} \to \mathbb{C} \cup \infty$ is surjective. If n is the sum of the orders of the poles of f in F, and $z_0 \in \mathbb{C}$, then $f(z) = z_0$ has n solutions (if a solution has multiplicity > 1, then it will count towards n multiple times).*

Proposition 4.1.1. *The values $\wp_\Gamma\left(\frac{\omega_i}{2}\right)$ are distinct for each ω_i.*

Proof. $\wp'(z)$ is doubly periodic, so $\wp'\left(\frac{\omega_i}{2}\right) = \wp'\left(-\frac{\omega_i}{2}\right)$. $\wp'$ is even, so $\wp'\left(-\frac{\omega_i}{2}\right) = -\wp'\left(\frac{\omega_i}{2}\right)$. Thus,

$$\wp'\left(\frac{\omega_i}{2}\right) = 0, \quad i = 1, 2, 3 \tag{12}$$

Therefore, each $\wp\left(\frac{\omega_i}{2}\right)$ is a root of $4x^3 - g_2x - g_3$. Let

$$h_i(z) = \wp(z) - \wp\left(\frac{\omega_i}{2}\right)$$

Then, $h_i\left(\frac{\omega_i}{2}\right) = h_i'\left(\frac{\omega_i}{2}\right) = 0$. In other words, h_i vanishes with order of vanishing at least 2 at the points $\frac{\omega_i}{2}$. Let F be the fundamental parallelogram of the lattice Γ on which $\wp$ is defined. Then, $h_i(z)$ has a single pole in F (the double pole at $z = 0$). By Theorem 4.1.1, we can deduce that $\frac{\omega_i}{2}$ is the only zero of $h_i(z)$. Particularly, $h_i\left(\frac{\omega_j}{2}\right)$ is nonzero when $j \neq i$. Therefore, each of $\wp\left(\frac{\omega_i}{2}\right)$ are distinct. □

Proposition 4.1.2. *Let $E : y^2 = f(x) = 4x^3 - g_2x - g_3$. Then,*

$$\Phi : \mathbb{C}/\Gamma \to E(\mathbb{C}) \tag{13}$$

$$z \mapsto (\wp_\Gamma(z), \wp_\Gamma'(z)) \tag{14}$$

$$0 \mapsto \infty \tag{15}$$

is an isomorphism of groups. That is, the elliptic curve E is isomorphic to the elliptic curve defined by C/Γ.

The proof of this proposition is not trivial. A sketch is provided below.

Proof. We begin by showing that Φ is surjective. Let $(x, y) \in E(\mathbb{C})$. The fuction $\wp(z) - x$ has a double pole, so it cannot be constant. Thus, it must have zeroes. Therefore, there exists $z \in \mathbb{C}$ such that $\wp(z) = x$. We have

$$\wp'(z) = y^2$$

so $\wp(z) = \pm y$. If $\wp'(z) = y$, then we have shown surjectivity, since we have found an element in C/Γ that maps to $(x, y) \in E(\mathbb{C})$ for every $(x, y) \in E(\mathbb{C})$. If $\wp'(z) = -y$, then since $\wp'(z)$ is an even function, $\wp'(-z) = y$, and $\wp(-z) = x$, so $-z \mapsto (x, y)$.
Next, we show that Φ is injective. Suppose $\wp(z_1) = \wp(z_2)$ and $\wp'(z_1) = \wp'(z_2)$ and z_1 and z_2 are not congruent modulo Γ (that is, they do not lie on the same lattice in $\mathbb{C}$). The poles of $\wp(z)$ all lie in Γ, so if z_1 is a pole for $\wp$, then $z_1 \in \Gamma$ and thus, $z_2 \in \Gamma$. But this would mean that z_1 and z_2 would be congruent modulo Γ. So, we may assume that z_1 is not a pole of $\wp$ and thus does not lie in Γ. Consider the function

$$h(z) = \wp(z) - \wp(z_1)$$

One can check that $h(z)$ has a double pole at $z = 0$ and no other poles in the fundamental parallelogram of Γ. By Theorem 4.1.1, $h(z)$ has exactly 2 zeroes. Suppose now that $z_1 = \frac{\omega_i}{2}$ for some i. Then, (12) tells us that $\wp'\left(\frac{\omega_i}{2}\right) = 0$, so z_1 is a double root of $h(z)$. Since we know that $h(z)$ has only two roots, z_1 is the only root of $h(z)$. We can conclude that $z_2 = z_1$. If z_1

does not take the form $\frac{\omega_i}{2}$, then since $h(-z_1) = h(z_1) = 0$, and since z_1 is not congruent to $-z_1$ modulo Γ, the two zeroes of $h(z)$ are z_1 and $-z_1$ modulo Γ. So, z_2 is congruent to $-z_1$ modulo Γ. However,

$$y = \wp'(z_2) = \wp'(-z_1) = -\wp'(z_1) = -y$$

so $\wp'(z_1) = y = 0$. However, $\wp'(z)$ has a triple pole, and so only three zeroes in F. (12) tells us that the three zeroes occur at $\frac{\omega_i}{2}$ for $i = 1, 2, 3$. But, $z \neq \frac{\omega_i}{2}$. This gives a contradiction, so z_1 is congruent to z_2 modulo Γ, and thus Φ is injective.
We have shown that Φ is bijective, so all that remains in to show that Φ is a group homomorphism. Once again, let $z_1, z_2 \in \mathbb{C}$. Let

$$\Phi(z_i) = P_i = (x_i, y_i)$$

We provide some restrictions on P_1 and P_2. Namely, assume that both P_1 and P_2 are finite and that the line that goes through P_1 and P_2 intersects E in three distinct finite points. Practically speaking this means that $P_1 \neq \pm P_2$, $P_1 + 2P_2 \neq \infty$, and $2P_1 + P_2 \neq \infty$. We can provide these restrictions without hindering the completeness of the proof because the addition formula on E is entirely different when $P_1 = P_2$ and the connection between double roots in algebraic calculation of points on E and double roots in the corresponding functions is not pertinent to proving the homomorphism. Let $y = ax + b$ be the line passing through P_1 and P_2. Then, by the constraints defined prior, $y = ax + b$ intersects E at a third distinct finite point. We will call this point P_3 and let

$$\Phi(z_3) = P_3 = (x_3, y_3)$$

where $z_3 \in \mathbb{C}$. Using the standard group law formulas for points on E, we have

$$\begin{aligned} x_3 &= \frac{1}{4}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2 \\ &= \frac{1}{4}\left(\frac{\wp'(z_2) - \wp'(z_1)}{\wp(z_2) - \wp(z_1)}\right)^2 - \wp(z_1) - \wp(z_2) \end{aligned}$$

Now, let

$$l(z) = \wp'(z) - a\wp(z) - b$$

$l(z)$ has zeroes at $z = z_1, z_2, z_3$. Since $l(z)$ has a triple pole at 0 and no other poles, it has three zeroes in F. We can, by utilizing tools from complex analysis, write

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_2) - \wp'(z_1)}{\wp(z_2) - \wp(z_1)}\right) - \wp(z_1) - \wp(z_2) \tag{16}$$

It can be shown that this formula holds for all z_i for which it is defined (recall that we excluded some z_i due to our constraints on points on E that we consider). We now turn our attention to the y-coordinate. We need to compute $\wp'(z_1 + z_2)$. Differentiating $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ yields

$$2\wp^{''}\wp' = (12\wp^2 - g_2)\wp' \tag{17}$$

We can divide by $\wp'$ to obtain

$$2\wp^{''}(z_2) = 12\wp(z_2)^2 - g_2$$

We can differentiate $\wp(z_1 + z_2)$ to get an expression $\wp'(z_1 + z_2)$ into which we can substitute (17). The resulting expression will give $\wp'(z_1 + z_2)$ in terms of $\wp(z_1)$, $\wp'(z_1)$, $\wp(z_2)$ and $\wp'(z_2)$. We can show that this expression is equivalent to $-y_3$. Therefore,

$$(\wp(z_1), \wp'(z_1)) + (\wp(z_2), \wp'(z_2)) = (\wp(z_1 + z_2), \wp'(z_1 + z_2))$$

In other words,

$$\Phi(z_1) + \Phi(z_2) = \Phi(z_1 + z_2) \tag{18}$$

However, we still have to check the cases of (18) when (16) is not defined. We can easily check the cases where $\wp(z_i) = \infty$ and z_1 is congruent to $-z_2$ modulo Γ. Suppose now that $z_1 = z_2$. Then, let $z_2 \to z_1$ and use (16) and (17) in conjunction with l'Hopital's rule to get

$$\begin{aligned}
\wp(2z_1) &= \frac{1}{4}\left(\frac{\wp''(z_1)}{\wp'(z_1)}\right)^2 - 2\wp(z_1) \\
&= \frac{1}{4}\left(\frac{6\wp(z_1)^2 - \frac{1}{2}g_2}{\wp'(z_2)}\right)^2 - 2\wp(z_1) \\
&= \frac{1}{4}\left(\frac{6x_1^2 - \frac{1}{2}g_2}{y_1}\right)^2 - 2x_1
\end{aligned}$$

This is the formula for x_3 we get from addition formula with P_1 and P_2. We can differentiate with respect to z_1 to obtain a similar formula to (18). Therefore, we have

$$\Phi(z_1) + \Phi(z_1) = \Phi(2z_1)$$

and we are done. [2] □

Note that we can deduce from this that the modular cusp form Δ is not identically 0, since its values are discriminants of nonsingular elliptic curves, as shown.

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