Homework 9

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Exercise 4.6 (Estimating American put prices.) *For each n, where $n = 0, 1, \ldots, N$, let G_n be a random variable depending on the first n coin tosses. The time-zero value of a derivative security that can be exercised at any time $n \leq N$ for payoff G_n but must be exercised at time N if it has not been exercised before that time is*

$$V_0 = \max_{\tau \in S_0, \tau \leq N} \tilde{E}\left[\frac{1}{(1+r)^\tau} G_\tau\right]$$

1. *Consider $G_n = K - S_n$, the derivative security that permits its owner to sell one share of stock for payment K at any time up to and including N, but if the owner does not sell by time N, then she must do so at time N. Show that the optimal exercise policy is to sell the stock at time zero and that the value of this derivative security is $K - S_0$. (assume $r \geq 0$.)*

   According to the algorithm described for determining the risk-neutral price of the American option, we should exercise at time 0 if $G_0 \geq V_0$. We know that $G_0 = K - S_0$. And we calculate V_0 using the risk-neutral formula below.

$$\begin{aligned} V_0 &= \frac{1}{1+r}\left[\tilde{p}(K - uS_0) + \tilde{q}(K - dS_0)\right] \\ &= \frac{1}{1+r}\left[\tilde{p}K - \tilde{p}uS_0 + \tilde{q}K - \tilde{q}dS_0\right] \\ &= \frac{1}{1+r}\left[K(\tilde{p} + \tilde{q}) - S_0(\tilde{p}u + \tilde{q}d)\right] \\ &= \frac{1}{1+r}\left[K - S_0(1+r)\right] \\ &= \frac{K}{1+r} - S_0 \end{aligned}$$

   And

$$G_0 = K - S_0 \geq \frac{K}{1+r} - S_0 = V_0$$

   so that we should always exercise at time 0.

2. *Explain why a portfolio that holds the derivative security in (1) and a European call with strike K and expiration time N is at least as valuable as an American put struck at K with expiration time N. Denote the time-zero value of the*

European call by V_0^{EC} *and the time-zero value of the American put by* V_0^{AP}. *Conclude that the upper bound*

$$V_0^{AP} \leq K - S_0 + V_0^{EC}$$

on V_0^{AP} *holds.*

If we look at the payoff function of the security in (1) and a European call, we have

$$K - S + \max\{S - K, 0\} = \begin{cases} 0 & S > K \\ K - S & S < K \end{cases}$$

which is exactly the payoff of the American put. In this case, we have a little more flexibility with the derivative in (1) so that the value of the American put should be at most the value of (1).

3. *Use put-call parity to derive the lower bound on* V_0^{AP} *:*

$$\frac{K}{(1+r)^N} - S_0 + V_0^{EC} \leq V_0^{AP}.$$

The put-call parity says

$$P + S = C + \frac{K}{(1+r)^N}$$

where P is a put option and C is a call option. So that the inequality here can be rewritten to fit into the put-call parity

$$\frac{K}{(1+r)^N} + V_0^{EC} \leq V_0^{AP} + S_0$$

Then we see that the American put should have a larger value since it has more flexibility than the the discounted cash plus the European call.