Homework 6

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Exercise 3.2 *Let P be a probability measure on a finite probability space Ω. In this problem, we allow the possibility that $P(\omega) = 0$ for some values of $\omega \in \Omega$. Let Z be a random variable on Ω with the property that $P(Z \geq 0) = 1$ and $EZ = 1$. For $\omega \in \Omega$, define $\tilde{P}(\omega) = Z(\omega)P(\omega)$, and for events $A \subset \Omega$, define $\tilde{P}(A) = \sum_{\omega\in\Omega} \tilde{P}(\omega)$. Show the following.*

1. *$\tilde{P}$ is a probability measure.*
   From the definitions
   $$\sum_{\omega\in\Omega} \tilde{P} = \sum_{\omega\in\Omega} Z(\omega)P(\omega) = EZ = 1.$$

2. *If Y is a random variable, the $\tilde{E}[Y] = E[ZY]$.*
   Again, from the definitions
   $$\begin{aligned}\tilde{E}[Y] &= \sum_{\omega\in\Omega} \tilde{P}(\omega)Y(\omega) = \sum_{\omega\in\Omega} \tilde{P}(\omega)Y(\omega)\frac{P(\omega)}{P(\omega)} \\ &= \sum_{\omega\in\Omega} P(\omega)Z(\omega)Y(\omega) = E[ZY].\end{aligned}$$

3. *If A is an event with $P(A) = 0$, then $\tilde{P}(A) = 0$.*
   From the definitions we have
   $$\tilde{P}(A) = \sum_{\omega\in A} \tilde{P}(\omega) = \sum_{\omega\in A} Z(\omega)P(\omega).$$
   Where we are given that for every $\omega \in A$, $P(A) = 0$, so that $\tilde{P}(A) = 0$.

4. *Assume that $P(Z > 0) = 1$. Show that if A is an event with $\tilde{P}(A) = 0$, then $P(A) = 0$.*
   As with the previous exercise
   $$P(A) = \sum_{\omega\in A} P(\omega) = \sum_{\omega\in A} \frac{\tilde{P}(\omega)}{Z(\omega)}$$
   Because $\tilde{P}(\omega) = 0$ for every $\omega \in A$ and $P(Z > 0) = 1$, $P(A) = 0$.

5. *Show that if P and $\tilde{P}$ are equivalent, then they agree which events have probability one.*

   First, assume $P(A) = 1$, then

$$\tilde{P}(A) = \sum_{\omega \in A} \tilde{P}(\omega) = \sum_{\omega \in A} Z(\omega) P(\omega) = E[Z] = 1.$$

   Next assume $\tilde{P}(A) = 1$, then

$$P(A) = 1 - P(A^C) = 1 - \sum_{\omega \in A^C} P(\omega) = 1 - \sum_{\omega \in A^C} \frac{\tilde{P}(\omega)}{Z(\omega)} = 1.$$

6. *Construct an example in which we have only $P(Z \geq 0) = 1$ and P and $\tilde{P}$ are not equivalent.*

   The following densities on three tosses demonstrates non-equivalent measures.

$$P = \begin{cases} 1/8 & \text{all heads} \\ 3/8 & \text{one tail} \\ 3/8 & \text{one head} \\ 1/8 & \text{all tails} \end{cases}$$

$$\tilde{P} = \begin{cases} 1/3 & \text{all heads} \\ 1/3 & \text{one tail} \\ 1/3 & \text{one head} \\ 0 & \text{all tails} \end{cases}$$

Exercise 3.4 *This problem refers to the model of Example 3.1.2, whos Radon-Nikodym process Z_n appears in Figure 3.2.1.*

1. *Compute the state price densities explicitly.*

   The state prices are given by the equation

$$\zeta_n(\omega) = \frac{Z(\omega)}{(1+r)^n}.$$

   The values for $Z(\omega)$ are given in Example 3.1.2, so that the density of the state prices is given as

$$\zeta_3 = \begin{cases} 27/125 & \omega = (HHH) \\ 54/125 & \omega = (THH, HTH, HHT) \\ 108/125 & \omega = (TTH, HTT, THT) \\ 216/125 & \omega = (TTT) \end{cases}$$

2. *Use the number computed above in formula (3.1.10) to find the time-zero price of the Asian option of Exercise 1.8 of Chapter 1.*

   Equation (3.1.10) is given as

$$V_0 = \sum_{\omega \in \Omega} V_N(\omega) \zeta(\omega) P(\omega),$$

the values for V_N, taken from the previous homework assignment are

$$V_3 = \begin{cases} 11 & \omega = (HHH) \\ 5 & \omega = (HHT) \\ 2 & \omega = (HTH) \\ 0.5 & \omega = (HTT) \\ 0.5 & \omega = (THH) \\ 0 & \omega = (THT) \\ 0 & \omega = (TTH) \\ 0 & \omega = (TTT) \end{cases}$$

and the values for $P(\omega)$ are given in the text. Then we have the following as the time-zero price :

$$\begin{aligned} V_0 &= 11\left(\frac{27}{125}\right)\frac{8}{27} + 5\left(\frac{54}{125}\right)\frac{4}{27} + 2\left(\frac{54}{125}\right)\frac{4}{27} + \frac{1}{2}\left(\frac{108}{125}\right)\frac{2}{27} + \frac{1}{2}\left(\frac{108}{125}\right)\frac{2}{27} \\ &= \frac{1}{125}(88 + 40 + 16 + 4 + 4) \\ &= 1.216 \end{aligned}$$

3. *Compute also the state densities* $\zeta_2(HT) = \zeta_2(TH)$.
From the text, we have

$$Z_2(HT) = Z_2(TH) = \frac{9}{8}$$

so that

$$\zeta_2(HT) = \zeta_2(TH) = \frac{9}{8}\left(\frac{4}{5}\right)^2 = \frac{18}{25}$$

4. *Use the risk-neutral pricing formula (3.2.6) in the form*

$$\begin{aligned} V_2(HT) &= \frac{1}{\zeta_2(HT)}E_2[\zeta_3 V_3](HT) \\ V_2(TH) &= \frac{1}{\zeta_2(TH)}E_2[\zeta_3 V_3](TH) \end{aligned}$$

to compute $V_2(HT)$ *and* $V_2(TH)$.
We begin with $V_2(HT)$, using values given in the previous part of this exercise, we have

$$\begin{aligned} V_2(HT) &= \frac{1}{\zeta_2(HT)}\left[P(H)\zeta_3(HTH)V_3(HTH) + P(T)\zeta_3(HTT)V_3(HTT)\right] \\ &= \frac{25}{18}\left(2\frac{2}{3}\frac{54}{125} + \frac{1}{2}\frac{1}{3}\frac{108}{125}\right) = 1 \end{aligned}$$

$$\begin{aligned} V_2(TH) &= \frac{1}{\zeta_2(TH)}\left[P(H)\zeta_3(THH)V_3(THH) + P(T)\zeta_3(THT)V_3(THT)\right] \\ V_2(TH) &= \frac{25}{18}\left(\frac{1}{2}\frac{2}{3}\frac{54}{125} + 0\frac{1}{3}\frac{108}{125}\right) = 0.2 \end{aligned}$$