Homework 4

Jeremy Morris

September 22, 2006

Exercise 2.9 (Stochastic volatilty, random interest rate) *Consdier a two-period stochastic volatility, random interest rate model of the type described in Exercise 1.9 of Chapter 1. The stock prices and interest rates are shown in Figure 2.8.1*

1. *Determine risk-neutral probabilities*

$$\tilde{P}(HH), \tilde{P}(HT), \tilde{P}(TH), \tilde{P}(TT)$$

such that the time-zero value of an option that pays off V_2 at time two is given by the risk-neutral pricing formula

$$V_0 = \tilde{E}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right]$$

We can use conditional probabilities to calculate the risk-neutral probabilities.

$$\begin{aligned}
\tilde{P}\{H_1H_2\} &= \tilde{P}\{H_1\}\tilde{P}\{H_2 \mid H_1\} \\
\tilde{P}\{H_1\} &= \frac{1+r_0-d_0}{u_0-d_0} = \frac{1+1/4-1/2}{2-1/2} = \frac{1}{2} \\
\tilde{P}\{H_2 \mid H_1\} &= \frac{1+r_1(H)-d_1}{u_1-d_1} = \frac{1+1/4-1}{3/2-1} = \frac{1}{2} \\
\tilde{P}\{H_1H_2\} &= \frac{1}{2}\frac{1}{2} = \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\tilde{P}\{T_2 \mid H_1\} &= \frac{u_1-1-r_1(H)}{u_1-d_1} = \frac{3/2-1-1/4}{3/2-1} = \frac{1}{4} \\
\tilde{P}\{H_1T_2\} &= \frac{1}{2}\frac{1}{2} = \frac{1}{4}
\end{aligned}$$

$$\tilde{P}\{T_1\} = \frac{u_0 - 1 - r_0}{u_0 - d_0} = \frac{2 - 1 - 1/4}{2 - 1/2} = \frac{1}{2}$$

$$\tilde{P}\{H_2 \mid T_1\} = \frac{1 + r_1(T) - d_1}{u_1 - d_1} = \frac{1 + 1/2 - 1}{4 - 1} = \frac{1}{6}$$

$$\tilde{P}\{T_1 H_2\} = \frac{1}{2}\frac{1}{6} = \frac{1}{12}$$

$$\tilde{P}\{T_2 \mid T_1\} = \frac{u_1 - 1 - r_1(T)}{u_1 - d_1} = \frac{4 - 1 - 1/2}{4 - 1} = \frac{5}{6}$$

$$\tilde{P}\{T_1 T_2\} = \frac{1}{2}\frac{5}{6} = \frac{5}{12}$$

2. *Let $V_2 = (S_2 - 7)^+$. Compute $V_0, V_1(H)$ and $V_2(T)$.*

$$V_1(H) = \frac{1}{1 + r_1(H)} \left(\tilde{P}\{HH\}V_2(HH) + \tilde{P}\{HT\}V_2(HT)\right)$$
$$= \frac{1}{1 + 1/4}\left(\frac{1}{4}5 + \frac{1}{4}1\right) = \frac{6}{5}$$

$$V_1(T) = \frac{1}{1 + r_1(T)} \left(\tilde{P}\{TH\}V_2(TH) + \tilde{P}\{TT\}V_2(TT)\right)$$
$$= \frac{1}{1 + 1/2}\left(\frac{1}{12}1 + \frac{5}{12}0\right) = \frac{1}{18}$$

$$V_0 = \frac{1}{1 + r_0} \left(\tilde{P}\{H\}V_1(H) + \tilde{P}\{T\}V_1(T)\right)$$
$$= \frac{1}{1 + 1/4}\left(\frac{1}{2}\frac{6}{5} + \frac{1}{2}\frac{1}{18}\right) = 0.50222$$

3. *Suppose an agent sells the option described above for V_0 at time zero. Compute the position Δ_0 she should take in the stock at time zero so that at time one, regardless of whether the first coin toss results in head or tail, the value of her portfolio is V_1.*

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$
$$= \frac{6/5 - 1/18}{8 - 2} = 0.1907407$$

4. *Suppose that the first coin toss results in head. What position $\Delta_1(H)$ should the agent now take in the stock to be sure that, regardless of the second toss, the value of her portfolio at time two will be $(S_2 - 7)^+$.*

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{5-1}{12-8} = 1$$

Exercise 2.10 (Dividend-paying stock) *An agent who begins with initial capital X_0 and at each time n takes a position of Δ_n shares of stock, where Δ_n depends only on the first n coin tosses, has a portfolio value governed by the wealth equation*

$$X_{n+1} = \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)$$

1. *Show that the discounted wealth process is a martingale under the risk-neutral measure.*

   The discounted wealth process is given as

$$\frac{X_n}{(1+r)^n} = \tilde{E}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right]$$

   Then we show that it holds for the particular form of the wealth equation given in this problem.

$$\begin{aligned}
\tilde{E}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{E}_n\left[\frac{\Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}\right] \\
&= \tilde{E}_n\left[\frac{\Delta_n Y_{n+1} S_n}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}}\tilde{E}_n\left[Y_{n+1}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n}
\end{aligned}$$

   Then all we have to do is show that $\tilde{E}_n\left[Y_{n+1}\right] = (1+r)$.

$$\begin{aligned}
\tilde{E}_n\left[Y_{n+1}\right] &= \tilde{p}u + \tilde{q}d \\
&= \frac{1+r-d}{u-d}u + \frac{u-1-r}{u-d}d \\
&= \frac{(1+r)-d}{u-d}u + \frac{u-(1+r)}{u-d}d \\
&= \frac{u(1+r)-ud}{u-d} + \frac{ud-d(1+r)}{u-d} \\
&= \frac{(u-d)(1+r)}{u-d} = (1+r)
\end{aligned}$$

   Then

$$\begin{aligned}
\tilde{E}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \frac{\Delta_n S_n}{(1+r)^{n+1}}(1+r) + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{\Delta_n S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{X_n}{(1+r)^n}
\end{aligned}$$

2. *Show that the risk-neutral pricing formula still applies.*

   This fact follows from the previous exercise and Theorem 1.2.2.

3. *Show that the discounted stock price is not a martingale under the risk-neutral measure. However, if A_{n+1} is a constant $a \in (0,1)$, regardless of the value of n and the outcome of the coin tossing $\omega_1 \ldots \omega_{n+1}$, then $\frac{S_n}{(1-a)^n(1+r)^n}$ is a martingale under the risk-neutral measure.*

   The stock price is given as follows

$$S_{n+1} = (1 - A_{n+1})Y_{n+1}S_n$$

   Then we have to show

$$\frac{S_n}{(1+r)^n} \neq \tilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]$$

$$\begin{aligned}
\tilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\tilde{E}_n\left[(1-A_{n+1})Y_{n+1}S_n\right] \\
&= \frac{S_n}{(1+r)^{n+1}}\tilde{E}_n\left[(1-A_{n+1})Y_{n+1}\right] \\
&= \frac{S_n}{(1+r)^n}\tilde{E}_n\left[(1-A_{n+1})\right] \\
&= \frac{S_n}{(1+r)^n} - \tilde{E}_n\left[A_{n+1}\right]
\end{aligned}$$

   If $a \in (0,1)$ is some constant and we take a different discounted stock price, this will become a martingale.

$$\frac{S_n}{(1-a)^n(1+r)^n} = \tilde{E}_n\left[\frac{S_{n+1}}{(1-a)^{n+1}(1+r)^{n+1}}\right]$$

$$\begin{aligned}
&= \frac{1}{(1-a)^{n+1}(1+r)^{n+1}}\tilde{E}_n\left[(1-a)Y_{n+1}S_n\right] \\
&= \frac{1}{(1-a)^{n+1}(1+r)^{n+1}}(1+r)(1-a)S_n \\
&= \frac{S_n}{(1-a)^n(1+r)^n}
\end{aligned}$$