Homework 3

Jeremy Morris

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Exercise 2.2 *Consider the stock price S_3 in Figure 2.3.1*

1. *What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = 1/2, \tilde{q} = 1/2$.*

$$
\begin{aligned}
P\{S_3 = 32\} &= \frac{1}{8}, \omega = (HHH) \\
P\{S_3 = 8\} &= \frac{3}{8}, \omega = (HHT, THH, HTH) \\
P\{S_3 = 2\} &= \frac{3}{8}, \omega = (TTH, HTT, THT) \\
P\{S_3 = 0.50\} &= \frac{1}{8}, \omega = (TTT)
\end{aligned}
$$

2. *Compute $\tilde{\mathbf{E}}S_1$, $\tilde{\mathbf{E}}S_2$ and $\tilde{\mathbf{E}}S_3$. What is the average rate of growth of the stock price under $\tilde{\mathbf{P}}$?*

$$
\begin{aligned}
\tilde{\mathbf{E}}S_1 &= 2P\{S_1 = 2\} + 8P\{S_1 = 8\} = 2(1/2) + 8(1/2) = 5 \\
\tilde{\mathbf{E}}S_2 &= 1P\{S_2 = 1\} + 4P\{S_2 = 4\} + 16\{S_2 = 16\} \\
&= 1(1/4) + 4(2/4) + 16(1/4) = 6.25 \\
\tilde{\mathbf{E}}S_3 &= 0.5P\{S_3 = 0.5\} + 2P\{S_3 = 2\} + 8\{S_3 = 8\} + 32\{S_3 = 32\} \\
&= 0.5(1/8) + 2(3/8) + 8(3/8) + 32(1/8) = 7.8125
\end{aligned}
$$

The average rate of growth δ is defined as

$$\delta = \frac{1}{n-1} \sum_{i=2}^{n} \frac{ES_i}{ES_{i-1}}$$

In this case we have

$$\delta = \frac{1}{2} \left(\frac{6.25}{5} + \frac{7.8125}{6.25} \right) = 1.25$$

3. *Answer (1) and (2) again under the actual probabilities $p = 2/3$, $q = 1/3$.*

The distribution from (1) would be

$$\begin{aligned}
P\{S_3 = 32\} &= \frac{8}{27}, \omega = (HHH) \\
P\{S_3 = 8\} &= \frac{12}{27}, \omega = (HHT, THH, HTH) \\
P\{S_3 = 2\} &= \frac{6}{27}, \omega = (TTH, HTT, THT) \\
P\{S_3 = 0.50\} &= \frac{1}{27}, \omega = (TTT)
\end{aligned}$$

And the expected values from (2) would be

$$\begin{aligned}
\mathbf{E}S_1 &= 2P\{S_1 = 2\} + 8P\{S_1 = 8\} = 2(1/3) + 8(2/3) = 6 \\
\mathbf{E}S_2 &= 1P\{S_2 = 1\} + 4P\{S_2 = 4\} + 16\{S_2 = 16\} \\
&= 1(1/9) + 4(5/9) + 16(4/9) = 9.444 \\
\mathbf{E}S_3 &= 0.5P\{S_3 = 0.5\} + 2P\{S_3 = 2\} + 8\{S_3 = 8\} + 32\{S_3 = 32\} \\
&= 0.5(1/27) + 2(6/27) + 8(12/27) + 32(8/27) = 13.5
\end{aligned}$$

The average rate of growth is

$$\delta = \frac{1}{2}\left(\frac{9.444}{6} + \frac{13.5}{9.444}\right) = 1.502$$

Exercise 2.4 *Toss a coin repeatedly. Assume the probability of head on each toss is 1/2, as is the probability of a tail. Let $X_j = 1$ if the j^{th} tail results in a head and $X_j = -1$ if it results in a tail. Consider the stochastic process $M_0, M_1, M_2, \ldots$ defined by $M_0 = 0$ and*

$$M_n = \sum_{j=1}^{n} X_j, n \geq 1$$

This is called a symmetric random walk.

1. *Using the properties of Thm 2.3.2, show that $M_0, M_1, M_2, \ldots$ is a martingale.*
   We can show this directly, the distribution of M_n is

$$M_n = \begin{cases} M_{n-1} + 1 & \text{if } \omega_n = H \\ M_{n-1} - 1 & \text{if } \omega_n = T \end{cases}$$

   Then we show that $E_n[M_{n+1}] = M_n$.

$$\begin{aligned}
E_n[M_{n+1}] = E\left[\sum_{i=0}^{n+1} X_i \mid \omega_1\omega_2\ldots\omega_n\right] &= E\left[M_{n-1} + \sum_{i=n}^{n+1} X_i \mid \omega_n\right] \\
= M_{n-1} + E\left[\sum_{i=n}^{n+1} X_i \mid \omega_n\right]
\end{aligned}$$

this proves that M_n is a Martingale since

$$E\left[\sum_{i=n}^{n+1} X_i \mid \omega_n\right] = \begin{cases} 1 & \text{if } \omega_n = H \\ -1 & \text{if } \omega_n = T \end{cases}$$

2. *Let σ be a positive constant and, for $n \geq 0$, define*

$$S_n = e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^n$$

Show that $S_0, S_1, S_2, \ldots$ is a martingale.

Here we use the same method as in the previous question. The distribution for S_n is

$$S_n = \begin{cases} S_{n-1}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)e^{\sigma} & \text{if } \omega_n = H \\ S_{n-1}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)e^{-\sigma} & \text{if } \omega_n = T \end{cases}$$

Then for the conditional expectation side of the equation, we get

$$\begin{aligned} E_n[S_{n+1}] &= E_n\left[\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1} \exp\left(\sigma\sum_{i=0}^{n+1} X_i\right)\right] \\ &= E_n\left[\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n-1} \exp\left(\sigma\sum_{i=0}^{n-1} X_i\right)\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\exp\left(\sigma X_{n+1}\right)\right] \\ &= \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n-1} \exp\left(\sigma\sum_{i=0}^{n-1} X_i\right) E_n\left[\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\exp\left(\sigma X_{n+1}\right)\right] \\ &= S_n \end{aligned}$$

Where

$$E_n\left[\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\exp\left(\sigma X_{n+1}\right)\right] = \begin{cases} \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)e^{\sigma} & \text{if } \omega_n = H \\ \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)e^{-\sigma} & \text{if } \omega_n = T \end{cases}$$

Exercise 2.5 *Let $M_0, M_1, \ldots$ be the symmetric random walk of Exercise 2.4, and define $I_0 = 0$ and*

$$I_n = \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j),\ n = 1, 2, \ldots$$

1. *Show that*

$$I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$$

To start we have

$$I_n = \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j) = \sum_{j=0}^{n-1} M_j M_{j+1} - M_j^2$$

Then we add zero

$$
\begin{aligned}
I_n &= \sum_{j=0}^{n-1} M_j M_{j+1} - M_j^2 + \frac{1}{2}M_{j+1}^2 - \frac{1}{2}M_{j+1}^2 + \frac{1}{2}M_j^2 - \frac{1}{2}M_j^2 \\
&= \sum_{j=0}^{n-1} -\frac{1}{2}(M_{j+1} - M_j)^2 + \frac{1}{2}M_{j+1}^2 - \frac{1}{2}M_j^2 \\
&= \frac{1}{2}\left(\sum_{j=0}^{n-1} M_{j+1}^2 - M_j^2\right) - \frac{n}{2} \\
&= \frac{1}{2}M_n - \frac{n}{2}
\end{aligned}
$$

where the sum in the last step is a telescoping series.

2. *Let n be an arbitrary nonnegative integer, and let $f(i)$ be an arbitrary function of a variable i. In terms of n and f, define another function $g(i)$ satisfying*

$$E_n[f(I_{n+1})] = g(I_n)$$

Here we just take the expectation

$$
\begin{aligned}
E_n\left[f(I_{n+1})\right] &= E_n\left[f\left(\sum_{j=0}^{n} M_j(M_{j+1} - M_j)\right)\right] \\
&= E_n\left[f(I_n) + f\left(M_n(M_{n+1} - M_n)\right)\right]
\end{aligned}
$$

Here we drop the function f, since it can be any arbitrary function, we take it to be $f(i) = i$.

$$
\begin{aligned}
E_n[I_{n+1}] &= E_n\left[I_n + (M_n(M_{n+1} - M_n)\right] \\
&= E_n\left[\frac{1}{2}M_n^2 - \frac{n}{2} + M_n M_{n+1} - M_n^2\right] \\
&= -\frac{1}{2}M_n^2 - \frac{n}{2} + E_n\left[M_n M_{n+1}\right] \\
&= -\frac{1}{2}M_n^2 - \frac{n}{2} + M_n E_n\left[M_{n+1}\right] \\
&(M_{n+1} \text{ is a Martingale}) \\
&= -\frac{1}{2}M_n^2 - \frac{n}{2} + M_n^2 \\
&= \frac{1}{2}M_n^2 - \frac{n}{2} \\
&= I_n
\end{aligned}
$$

So that $g(I_n) = I_n$.