Homework 2

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Exercise 1.8 (Asian option) *Consider the three-period model of Example 1.2.1, with $S_0 = 4, u = 2, d = 1/2$, and take the interest rate $r = 1/4$, so that $\tilde{p} = \tilde{q} = 1/2$. For $n = 0, 1, 2, 3$, define $Y_n = \sum_{k=0}^{n} S_k$. Consider and Asian call option that expires at time three and has strike $K = 4$. This is like a European call, except the payoff of the option is based on the average stock price rather than the final stock price. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$. In particular, $v_3(s, y) = (\frac{1}{4}y - 4)^+$.*

1. *Develop an algorithm for computing v_n recursively. In particular, write a formula for v_n in terms of v_{n+1}.*

$$v_n(s, y) = \frac{1}{1+r}\left[\tilde{p}v_{n+1}\left(2s, y+2s\right) + \tilde{q}v_{n+1}\left(\frac{1}{2}s, y+\frac{1}{2}s\right)\right]$$

2. *Apply the algorithm developed in (1) to compute $v_0(4, 4)$, the price of the Asian option at time 0.*

   I wrote a perl program to calculate this recursively, the answer it calculated was 1.216. (Code available on request.)

3. *Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$.*

$$\begin{aligned}
\delta_n(s, y) &= \frac{v_{n+1}(2s, y+2s) - v_{n+1}(\frac{1}{2}s, y+\frac{1}{2}s)}{2(n+1)s - \frac{1}{2}(n+1)s} \\
&= \frac{v_{n+1}(2s, y+2s) - v_{n+1}(\frac{1}{2}s, y+\frac{1}{2}s)}{\frac{3}{2}(n+1)s} \\
&= \frac{2}{3}\frac{v_n(s, y)}{(n+1)s}
\end{aligned}$$

Exercise 1.9 (Stochastic volatility, random interest rate) *Consider a binomial pricing model, but at each time $n \geq 1$, the "up factor" u_n, the down factor d_n and the interest rate r_n are allowed to depend on n and on the first n coin tosses $\omega_1\omega_2\ldots\omega_n$. The initial up factor u_0, the initial down factor d_0 and the initial interest rate r_0 are not random. More specifically, the stock price at time one is given by*

$$S_1(\omega_1) = \begin{cases} u_0 S_0 & \text{if } \omega_1 = H \\ d_0 S_0 & \text{if } \omega_1 = T \end{cases}$$

and, for $n \geq 1$, the stock price at time $n+1$ is given by

$$S_{n+1}(\omega_1\omega_2\ldots\omega_{n+1}) = \begin{cases} u_n(\omega_1\omega_2\ldots\omega_{n+1})S_n(\omega_1\omega_2\ldots\omega_{n+1}) & \text{if } \omega_{n+1} = H \\ d_n(\omega_1\omega_2\ldots\omega_{n+1})S_n(\omega_1\omega_2\ldots\omega_{n+1}) & \text{if } \omega_{n+1} = T \end{cases}$$

One dollar invested in or borrowed from the money market at time zero grows to an investment or debt of $1+r_0$ at time one, and , for $n \geq 1$ one dollar invested in or borrowed from the money market at time n grows to an investment or debt of $1+r_n(\omega_1\omega_2\ldots\omega_{n+1})$ at time $n+1$. We assume that for each n and for all $\omega_1\omega_2\ldots\omega_{n+1}$, the no-arbitrage condition holds.

1. *Let N be a positive integer. In the model just described, provide an algorithm for determining the price at time zero for a derivative security that at time N pays off a random amount V_N depending on the result of the first N coin tosses.*

   Here we use the same form of the recursive equation used above.

$$\begin{aligned} V_n &= \frac{1}{1+r_{n+1}}\left[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right] \\ &= \frac{1}{1+r_{n+1}}\left[\left(\frac{1+r_{n+1}-d_{n+1}}{u_{n+1}-d_{n+1}}\right)V_{n+1}(H) + \left(\frac{u_{n+1}-1-r_{n+1}}{u_{n+1}-d_{n+1}}\right)V_{n+1}(T)\right] \end{aligned}$$

   Where we have dropped the ω notation for readabilty.

2. *Provide a formula for the number of shares of stock that should be held at each time n by a portfolio that replicates the derivative security V_n.*

   Again, this formula follows the same format as in the previous question.

$$\begin{aligned} \Delta_n &= \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \\ &= \frac{V_n(\omega_1\omega_2\ldots\omega_n H) - V_n(\omega_1\omega_2\ldots\omega_n T)}{S_n(\omega_1\omega_2\ldots\omega_n H) - S_n(\omega_1\omega_2\ldots\omega_n T)} \end{aligned}$$

3. *Suppose the initial stock price is $S_0 = 80$, with each head the stock price increases by 10, and with each tail the stock price decreases by 10. Assume the interest rate is always zero. Consider a European call with the strike price $K = 80$, expiring at time five. What is the price of this call at time zero?*

   I wrote another perl program (altered the one from the first question) to come up with the answer 9.375. (Code available on request.)