Homework # 11

Jeremy Morris

November 20, 2006

1. *Let the current stock price be denoted*

$$S_n = S_0 \exp\left((r - \frac{1}{2}\sigma^2)n\Delta t + \sigma\sqrt{\Delta t}M_n\right)$$

where the random walk is given by

$$M_n = \sum_{j=1}^{n} X_j$$

with $M_0 = 0$. Each walk X_j is independent from the others and

$$X_j = \begin{cases} 1 & \omega_j = H \\ -1 & \omega_j = T \end{cases}$$

(a) *Fix time $T = N\Delta t$ and let $N \to \infty$, $\Delta t \to 0$. Using the central limit theorem, show that the distribution of $\log S_N/S_0$ converges to a normal distribution with mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$.*

We note here that

$$\log\frac{S_N}{S_0} = (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T/N}M_N$$

and that all values are fixed except M_N which is random. We know that by the central limit theorem M_N/N is normally distributed with some mean and variance. The mean is given as

$$E\left[\frac{1}{N}M_N\right] = \frac{1}{N}E\left[M_N\right] = \frac{1}{N}\sum_{j=1}^{N}E\left[X_j\right] = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

and the variance is given by

$$\text{var}\left(\frac{1}{N}M_N\right) = \left(\frac{1}{N}\right)^2\left(E[M_N^2] - E[M_N]^2\right)$$

we know that $E[M_N] = 0$ so that we are left with

$$\text{var}\left(\frac{1}{N}M_N\right) = \left(\frac{1}{N}\right)^2 E[M_N^2].$$

So we have

$$
\begin{aligned}
E[M_N^2] = E\left[\left(\sum_{i=1}^{N} X_i\right)^2\right] &= E\left[\sum_{i=1}^{N} X_i^2 + 2\sum_{i=1}^{N}\sum_{j\neq i} X_i X_j\right] \\
&= E\left[\sum_{i=1}^{N} X_i^2\right] + 2E\left[\sum_{i=1}^{N}\sum_{j\neq i} X_i X_j\right] \\
&= N + 0.
\end{aligned}
$$

So that $\frac{1}{N}M_N \sim N(0, 1/N)$. We will make the remark that if $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(\mu + b, a^2\sigma^2)$. This means that $M_N \sim N(0, N)$. From this fact, we also get that $\log S_N/S_0$ is normally distributed with the correct mean and variance specified in the question.

(b) *Show that if $X \sim N(\mu, \beta^2)$, the $E[e^X] = e^{\mu+\beta^2/2}$. Does e^X have a normal distribution?*

$E[e^X] = e^{\mu+\beta^2/2}$ is the definition of the moment generating function of the normal distribution. e^X has the log-normal distribution and only takes positive values.

(c) *Using part (b), assume T is fixed, show that*

$$
E\left[\lim_{\Delta t\to\infty} S(T)\right] = S_0 e^{rT}
$$

Then we have

$$
\begin{aligned}
E\left[\lim_{\Delta t\to\infty} S(T)\right] &= E\left[\lim_{\Delta t\to 0} S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{\Delta t}M_N}\right] \\
&= S_0 E\left[\lim_{\Delta t\to 0} e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{\Delta t}M_N}\right]
\end{aligned}
$$

From part (a), we have that, in the limit as $N \to \infty$ (or $\Delta t \to 0$), we have that $(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T/N}M_N \sim N((r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$. And from part(b), we get that taking the expectation gives us

$$
S_0 E\left[\lim_{\Delta t\to\infty} e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{\Delta t}M_N}\right] = S_0 e^{(r-\frac{1}{2}\sigma^2)T-\frac{1}{2}\sigma^2 T} = S_0 e^{rT}
$$

2. *Let S_n be the above process, show that the process*

$$
\tilde{S}_n = \frac{S_n}{D^n}
$$

where

$$
D = \frac{1}{2}(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{(r-\frac{1}{2}\sigma^2)\Delta t}
$$

is a martingale. When Δt is small enough show that

$$D \approx e^{r\Delta t} \approx (1 + r\Delta t).$$

We show from the martingale property that

$$\tilde{E}_n\left[\tilde{S}_{n+1}\right] = \tilde{S}_n.$$

To begin we have

$$\begin{aligned}
\tilde{E}_n\left[\tilde{S}_{n+1}\right] &= \tilde{E}_n\left[\frac{S_{n+1}}{D^{n+1}}\right] = \tilde{E}\left[\frac{S_0 e^{(r-\frac{1}{2}\sigma^2)(n+1)\Delta t+\sigma\sqrt{\Delta t}M_{n+1}}}{(\frac{1}{2}(e^{\sigma\sqrt{\Delta t}}+e^{-\sigma\sqrt{\Delta t}})e^{(r-\frac{1}{2}\sigma^2)\Delta t})^{n+1}}\right] \\
&= \frac{S_0 e^{(r-\frac{1}{2}\sigma^2)n\Delta t} e^{(r-\frac{1}{2}\sigma^2)\Delta t} e^{\sigma\sqrt{\Delta t}M_n}}{\left(\frac{1}{2}(e^{\sigma\sqrt{\Delta t}}+e^{-\sigma\sqrt{\Delta t}})e^{(r-\frac{1}{2}\sigma^2)\Delta t}\right)^{n+1}} \tilde{E}_n\left[e^{\sigma\sqrt{\Delta t}X_{n+1}}\right] \\
&= \frac{\left(S_0 e^{(r-\frac{1}{2}\sigma^2)n\Delta t+\sigma\sqrt{\Delta t}M_n}\right)\frac{1}{2}(e^{\sigma\sqrt{\Delta t}}+e^{-\sigma\sqrt{\Delta t}})e^{(r-\frac{1}{2}\sigma^2)\Delta t}}{\left(\frac{1}{2}(e^{\sigma\sqrt{\Delta t}}+e^{-\sigma\sqrt{\Delta t}})e^{(r-\frac{1}{2}\sigma^2)\Delta t}\right)^{n+1}} \\
&= \frac{\left(S_0 e^{(r-\frac{1}{2}\sigma^2)n\Delta t+\sigma\sqrt{\Delta t}M_n}\right)}{\left(\frac{1}{2}(e^{\sigma\sqrt{\Delta t}}+e^{-\sigma\sqrt{\Delta t}})e^{(r-\frac{1}{2}\sigma^2)\Delta t}\right)^{n}} = \tilde{S}_n
\end{aligned}$$