Final Project

Jeremy Morris

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1. *Use plain language, with simple numerical examples, to explain to your non-mathematical friends why the price of a call option on a stock does not depend on the expected growth rate of the stock price itself. The explanation should be typed and not exceed one page.*

   The short answer is that the price of a call option does not depend on the expected growth of the stock price because of arbitrage. Consider the situation where the seller of a call option bases the price of the option on an assumption that the stock will decrease in value. If the purchaser of the option has some inside information on the stock and knows that the value is more likely to increase, an arbitrage situation has been created where the purchaser can generate money from nothing. However, if the seller of the option bases the price off of a risk-neutral pricing structure, an arbitrage situation will not occur and the price will be fair.

2. *To make our binomial model more practical, we will need to extend it to accommodate time dependent (deterministic) risk-free rate and volatility, that is, we need to deal with different r_i and σ_i for $i = 0, ..., N-1$. The deterministic variable rate is rather easy to implement, we just need time-dependent risk-neutral probabilities $\tilde{p}_i$, $\tilde{q}_i$ with proper dependence on r_i. However, time-dependent volatility is a different matter, since we require the binomial tree to recombine. One way to achieve this is to allow Δt to vary by requiring*

$$e^{\sigma_i\sqrt{\Delta t_i}} * e^{\sigma_{i+1}\sqrt{\Delta t_{i+1}}} = e^{\sigma_i\sqrt{\Delta t_i}} * e^{\sigma_{i+1}\sqrt{\Delta t_{i+1}}},$$

   or

$$\sigma_i\sqrt{\Delta t_i} = \alpha, \textit{for } i = 0, ..., N-1$$

   for some constant $\alpha > 0$. Implement this extended model and use it to price an European call option with expiration $T = 0.25$ and strike $K = 50$. We assume that the current stock price $S_0 = 49$, and the following risk-free rate and volatility structures

$$r = \begin{cases} 5\% & 0 \le t < 0.125 \\ 4.5\% & 0.125 \le t < 0.25 \end{cases} \quad \sigma = \begin{cases} 25\% & 0 \le t < 0.125 \\ 20\% & 0.125 \le t < 0.25 \end{cases}$$

   We suggest $\alpha = 0.01$ as a reasonable choice for this computation. If we switch the volatility structure so that

$$\sigma = \begin{cases} 20\% & 0 \le t < 0.125 \\ 25\% & 0.125 \le t < 0.25 \end{cases}$$

What is the call option price? Can you give an interpretation for this increase or decrease, compared to the price corresponding to the previous volatility structure?

In the first case, we get the price $V_0 = 2.015006$. In the second we get $V_0 = 2.011331$. There is a difference of 0.003675115 between the two prices where the first scenario yields a higher price. There seems to be no reason for the difference. The stock prices are all the same. The only difference is in the number of time steps that get discounted at different $\tilde{p}, \tilde{q}$ and r. In the end, the difference is not very large.

3. *The payoff of the American digital call option is quite simple: if the stock hits K before the expiration T , the holder receives \$1 at that time. If the stock never hit K before T, the option holder receives nothing. From the holder's point of view, there is no agonizing decision to make because you cannot get more than \$1, so you might as well exercise it the first time the option is in the money. Price this American digital call using the same binomial model and the same parameters as in Project No.3.*

   The price for this option is 0.3000959. (Code is attached.)

4. *Exercise 6.4 from Shreve's textbook.*

   (a) *Determine $V_1(H)$ and $V_1(T)$, the price at time one of the caplet in the events $\omega_1 = H$ and $\omega_2 = T$, respectively.*

   The values are as follows :

$$V_1(H) = \frac{1}{1+R_1(H)}(pV_2(HH) + qV_2(HT)) = \frac{4}{21}$$
$$V_1(T) = \frac{1}{1+R_1(T)}(pV_2(TH) + qV_2(TT)) = 0$$

   (b) *Show how to begin with $2/21$ at time zero and invest in the money market and the maturity two bond in order to have a portfolio value X_1 at time one that agrees with V_1, regardless of the outcome of the first coin toss. Why do we invest in the maturity two bond rather than the maturity three bond to do this?*

   In order for the value to match, we need to solve the equation

$$X_1 = \Delta_{0,1}B_{0,1} + \Delta_{0,2}B_{1,2} + (1+R_0)(X_0 - \Delta_{0,2}B_{0,2})$$

   for $\Delta_{0,1}$ and $\Delta_{0,2}$. Then we need to buy $\Delta_{0,2} = 1.333$ bonds and put nothing in the money market. We use the maturity two bond to simulate the money market.

   (c) *Show how to take the portfolio value X_1 at time one and invest in the money market and the maturity three bond in order to have a portfolio value X_2 at time two that agrees with V_2, regardless of the outcome of the first two coin tosses. Why do we invest in the maturity three bond rather than the maturity two bond to do this?*

   In this case, we need to solve the equation

$$X_2 = \Delta_{1,2}B_{1,2} + \Delta_{1,3}B_{2,3} + (1+R_1)(X_1 - \Delta_{1,2}B_{1,2} - \Delta_{1,3}B_{1,3})$$

for $\Delta_{1,2}$ (the money market amount) and $\Delta_{1,3}$ (the bond amount). Then we get that $\Delta_{1,2} = 0.2659$ and $\Delta_{1,3} = 0.1329$. As in the previous exercise, we use the maturity three bond to simulate the money market.

5. *Implement the Ho-Lee model (page 165) for the short interest rate in a binomial tree, with our notation:*

$$R_{i+1} = R_i + a_i \Delta t + \sigma\sqrt{\Delta t} X_{i+1}, i = 0, ..., N-1$$

for $T = 5$, $N = 20$. We assume $R_0 = 5\%, a_i = -0.002, \sigma = 0.005$.

(a) *Use the obtained binomial model to price a 3-year interest cap with a cap rate $R_K = 5\%$ and a 3-month ($\Delta t = 0.25$) tenure.*

This cap has time zero value 0.03037294. (Code is attached.)