Jeremy Morris

Math 6070 Midterm (deadline: 23 April)

1. *Let $U_1, \ldots, U_n$ be independent identically distributed random variables, uniform on $[0,1]$. Let $U_{1,n} \leq U_{2,n} \leq \cdots \leq U_{n,n}$ denote the order statistics. Let k_n satisfy*

$$k_n \to \infty \text{ and } k_n/n \to p \in (0,1) \quad (n \to \infty).$$

Show that there are a_n, b_n such that

$$(a_n U_{k_n,n} - b_n) \xrightarrow{d} N(0,1).$$

We know that

$$(U_{1,n}, U_{2,n}, \ldots, U_{n,n}) \stackrel{d}{=} \left(\frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}} \right)$$

where

$$S_k = \sum_{i=1}^{k} Y_i.$$

and the Y_i are indpendent identically distributed exponential random variables with parameter 1. I will show that

$$\sqrt{n}\left(\frac{S_{k_n}}{S_{n+1}} - \frac{k_n}{n+1}\right) \xrightarrow{d} N(0,1). \tag{1}$$

Then we can do some algebraic manipulation so that (1) can be rewritten

$$\sqrt{n}\left(\frac{(n+1)S_{k_n} - k_n S_{n+1}}{(n+1)S_{n+1}}\right)$$

which becomes

$$\sqrt{n}\left(\frac{(n+1)(S_{k_n} - k_n) - k_n(S_{n+1} - (n+1))}{(n+1)S_{n+1}}\right)$$

and finally

$$\frac{(n+1)n}{(n+1)S_{n+1}}\left(\frac{1}{\sqrt{n}}(S_{k_n} - k_n) - \frac{1}{\sqrt{n}}\frac{k_n}{n+1}(S_{n+1} - (n+1))\right).$$

The first term in this statement converges in probability to one since

$$ES_{n+1} = n+1.$$

Then, by application of Slutsky's theorem, we need to show that the remaining term converges in distribution to the normal distribution. The remaining term is

$$\frac{1}{\sqrt{n}}(S_{k_n} - k_n) - \frac{1}{\sqrt{n}}\frac{k_n}{n+1}(S_{n+1} - (n+1)) \tag{2}$$

where S_{k_n} and S_{n+1} are not independent. In order to create a sum of independent random variables, we rewrite (2) so that we have

$$\frac{1}{\sqrt{n}}(S_{k_n} - k_n) - \frac{1}{\sqrt{n}}\frac{k_n}{n+1}(S_{n+1} - (n+1) - S_{k_n} + S_{k_n} + k_n - k_n)$$

which can be rewritten

$$\frac{1}{\sqrt{n}}(S_{k_n} - k_n) - \frac{1}{\sqrt{n}}\frac{k_n}{n+1}\Big[\Big(S_{k_n} - k_n\Big) + \Big(S_{n+1} - S_{k_n} - ((n+1) - k_n)\Big)\Big]. \qquad (3)$$

Then S_{k_n} and $(S_{n+1} - S_{k_n})$ are independent so that

$$\Big(\frac{1}{\sqrt{k_n}}(S_{k_n} - k_n), \frac{1}{\sqrt{n - k_n}}(S_{n+1} - S_{k_n} - ((n+1) - k_n)\Big) \xrightarrow{d} (N_1, N_2)$$

where N_1 and N_2 are independent standard normal random variables. Then as $n \to \infty$, (3) becomes

$$\sqrt{p}N_1 - p\Big[\sqrt{p}N_1 + \sqrt{1-p}N_2\Big],$$

which is a linear combination of independent normal random variables with mean 0 and variance $p(1-p)$. This shows that

$$(a_n U_{k_n,n} - b_n) \xrightarrow{d} N(0,1).$$

where $a_n = \sqrt{n}$ and $b_n = \sqrt{n}k_n/(n+1)$.