Midterm 2

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1. Find conditions on $f(\mathbf{x}, \theta)$ such that

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\log f(\mathbf{x}, \theta)\right)$$

holds.

2. Let f and g be two densities. Show that

$$\int_{-\infty}^{\infty} f(x)\log f(x)dx \geq \int_{-\infty}^{\infty} f(x)\log g(x)dx,$$

assuming that both integrals exist. When do we have equality?

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d.r.v.'s, uniform on $[0, \theta]$ $(\theta > 0)$.
(a) Find the maximum likelihood estimator for θ.
(b) Get the asymptotic distribution of the maximum likelihood estimator.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with $P\{X_1 = 1\} = p$ and $P\{X_1 = 0\} = 1 - p$. We wish to test $H_0 :\ p = p_0$ against $H_A :\ p \neq p_0$. Use the likelihood ratio test and get its asymptotic distribution directly.

5. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables uniform on $[0, \theta]$. We wish to test $H_0 :\ \theta = \theta_0$ against $H_A :\ \theta \neq \theta_0$. Use the likelihood ratio test and get its asymptotic distribution.

6. $X_1, X_2, \ldots, X_n$ be independent identically distributed $N(\mu, \sigma^2)$ random variables. We wish to test $H_0 :\ \mu = \mu_0$ against the alternative that $H_A :\ \mu \neq \mu_0$. Show that the t–test and the likelihood ratio test are equivalent.

7. $X_1, X_2, \ldots, X_n$ be independent identically distributed $N(\mu, \sigma^2)$ random variables. We wish to test $H_0 : \mu \leq \mu_0$ against the alternative $H_A : \mu > \mu_0$. Find a test using the likelihood ratio.

8. $X_1, X_2, \ldots, X_n$ be independent random variables. We assume that X_i is an exponential(θ_i random variable. We wish to test $\theta_1 = \theta_2 = \ldots = \theta_n$ against the alternative that the null is not true.
(a) Find a test using the likelihood ratio.
(b) Find the asymptotic distribution of the likelihood ratio test.

9. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed $N(\mu_1, \sigma^2)$ random variables. Let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed $N(\mu_2, \sigma^2)$. We wish to test $H_0: \quad \mu_1 = \mu_2$ against the alternative $H_A: \quad \mu_1 \neq \mu_2$. Show that the likelihood ratio test is equivalent with the two sample t–test.

10. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed $N(\mu_1, \sigma_1^2)$ random variables. Let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed $N(\mu_2, \sigma_2^2)$. We wish to test $H_0: \mu_1 = \mu_2$ against the alternative $H_A: \quad \mu_1 \neq \mu_2$. Find a test using the generalized likelihood.

11. We wish to test the null hypothesis that $\theta = \theta_0$ against the alternative that the null is not true . We reject the null if

$$\frac{|\ell_n'(\theta_0)|}{\sqrt{-\ell_n''(\theta_0)}} \text{ is large.}$$

Determine the asymptotic rejection region.

12. Let $X_1, X_2, \ldots$ be independent identically distributed normal $N(\mu, \sigma^2)$ random variables. We wish to have an $1 - \alpha$ confidence interval for μ of length not greater than d. Use sequential method to achieve this goal.