Exam 1 Answers

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Exercises 6b

2. *Given the model $Y_i = \beta_1 x_i + \varepsilon_i$ where $\varepsilon_i \sim N(0, \sigma^2 w_i^{-1})$ show how to predict x_0 for a given value of Y_0. Describe a method for constructing a confidence interval for x_0.*

   Prediction of x_0 can be done using the equation $\tilde{x} = Y_0/\tilde{\beta}_1$ where

   $$\tilde{\beta}_1 = \frac{\sum w_i Y_i (x_i - \bar{x}_w)}{\sum w_i (x_i - \bar{x}_w)^2}$$

   $$\bar{x}_w = \frac{\sum w_i x_i}{\sum w_i}$$

   A confidence interval for x_0 can be found by using the roots of

   $$x^2 \left(\tilde{\beta}_1^2 - \frac{\lambda^2 S^2}{\sum x_i^2} \right) - 2x\tilde{\beta}_1 Y_0 + Y_0^2 - \lambda^2 S^2 = 0$$

   where we take

   $$S^2 = \frac{1}{n-1} \left\{ \sum w_i Y_i^2 - \tilde{\beta}_1^{\,2} \sum w_i x_i^2 \right\}$$

3. *Given the regression line*

   $$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

   where the ε_i are independent with $E[\varepsilon_i] = 0$ and $var(\varepsilon_i) = \sigma^2 x_i^2$, show that weighted least squares estimation is equivalent to ordinary least squares estimation fo the model

   $$\frac{Y_i}{x_i} = \beta_1 + \frac{\beta_0}{x_i} + \delta_i$$

   The main idea is to use the weighted least squares method and set $w_i = x_i^{-2}$. Then we have

   $$\sum_{i=1}^{n} x_i^{-2} \left(Y_i - \beta_0 - \beta_1 x_i \right)^2 = \min$$

   This can be rewritten as

   $$\sum_{i=1}^{n} \left(\frac{Y_i - \beta_0 - \beta_1 x_i}{x_i} \right)^2 = \min$$

Which is equivalent to the model

$$\frac{Y_i}{x_i} = \beta_1 + \frac{\beta_0}{x_i} + \delta_i$$

Where $\delta_i = \varepsilon_i / x_i$.

Miscellaneous Exercises 6

2. *Derive an F-statistic for testing the hypothesis that two straight lines intersect at the point (a,b)*

   Define the combined model as $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. Where $\boldsymbol{Y}' = (y_{1,i}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,m})$ and

$$\boldsymbol{X} = \begin{pmatrix} 1 & x_{1,1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1,n} & 0 & 0 \\ 0 & 0 & 1 & x_{2,1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & x_{2,m} \end{pmatrix} \tag{1}$$

   And $\boldsymbol{\beta}' = (\alpha_1, \beta_1, \alpha_2, \beta_2)$. Then we use the F-statistic derived previously

$$F = \frac{(RSS_H - RSS)/q}{RSS/(n-p)} \tag{2}$$

   With

$$RSS_H - RSS = (A\hat{\beta} - c)\left(A(X'X)^{-1}A'\right)^{-1}(A\hat{\beta} - c)' \tag{3}$$

   And we set

$$A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \tag{4}$$

$$c = \begin{pmatrix} b \\ b \end{pmatrix} \tag{5}$$

3. *Obtain an estimate and a confidence interval for the horizontal distance between two parallel lines.*

   Since the lines are parallel we have the models

$$Y_1 = \alpha_1 + \beta_1 X_1 \tag{6}$$

$$Y_2 = \alpha_2 + \beta_1 X_2 \tag{7}$$

   Setting $Y_1 = Y_2 = 0$ and solving, we get the estimate

$$\delta = \frac{\hat{\alpha}_2 - \hat{\alpha}_1}{\hat{\beta}_1} \tag{8}$$

where we choose the models so that the estimate δ is positive. Then to get a confidence interval we define

$$U = \hat{\alpha}_2 - \hat{\alpha}_1 - \hat{\beta}_1 \delta \tag{9}$$

Then we have the following properties of U

$$\begin{aligned} EU &= 0 & (10)\\ var(U) &= var(\bar{Y}_2) + var(\bar{Y}_1) + (\bar{x}_1 - \bar{x}_2 - \delta)^2 var(\hat{\beta}_1) & (11)\\ &= \sigma^2 \left(\frac{1}{n_2} + \frac{1}{n_1} + \frac{(\bar{x}_1 - \bar{x}_2 - \delta)^2}{\sum\sum(x_{ik} - \bar{x}_{i\cdot})} \right) & (12)\\ &= \sigma_U^2 & (13) \end{aligned}$$

Where the covariance terms in $var(U)$ are all 0. Since U is a linear combination of the Y_i, we can say that U has the distribution $N(0, w\sigma^2)$ and we can construct the T variable

$$T = \frac{U/\sigma_U}{S/\sigma} \tag{14}$$

Where $S^2 = RSS_{H_1}/(n_1 + n_2 - 3)$ for the alternative hypothesis H_1 that the lines are parallel. Then the confidence interval are the roots in δ of the equation

$$T^2 = F^{\alpha}_{1, n_1 + n_2 - 3} \tag{15}$$

Miscellaneous Exercises 7

3. *Suppose that the regression curve*

$$E[Y] = \beta_0 + \beta_1 x + \beta_2 x^2$$

has a local maximum at $x = x_m$ where x_m is near the origin. if Y is observed at n points x_i in $[-a, a], \bar{x} = 0$, and the usual normality assumptions hold, outline a method for finding a confidence interval for x_m.

First find a statistic for $\hat{x}_m$ by differentiating and solving for 0 to find the value $\hat{x}_m$

$$\begin{aligned} 0 &= \beta_1 + 2\beta_2 x & (16)\\ \hat{x}_m &= \frac{-\beta_1}{2\beta_2} & (17) \end{aligned}$$

Then let $U = \hat{\beta}_1 + 2\hat{\beta}_2 x_m$ which has the properties

$$\begin{aligned} E[U] &= 0 & (18)\\ var(U) &= var(\hat{\beta}_1) + 4x_m^2 var(\hat{\beta}_2) + 4x_m^2 cov(\hat{\beta}_2, \hat{\beta}_1) = \sigma_U^2 & (19) \end{aligned}$$

Where $var(\hat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$. Which means we have the new variable

$$T = \frac{U/\sigma_U}{S/\sigma} \sim t_{n-3} \tag{20}$$

And the confidence interval can be defined by the roots of the quadratic

$$T^2 = F^{\alpha}_{1,n-3} \tag{21}$$

in x_m.

Miscellaneous Exercises 9

1. *Suppose that the postulated regression model is*

$$EY = \beta_0 + \beta_1 x$$

when the true model is

$$EY = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$

If the observations of EY are taken at $x = -3, -2, -1, 0, 1, 2, 3$ what is the bias?

If we let

$$X = \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \tag{22}$$

$$Z = \begin{pmatrix} 9 & -27 \\ 4 & -8 \\ 1 & -1 \\ 0 & 0 \\ 1 & 1 \\ 4 & 8 \\ 9 & 27 \end{pmatrix} \tag{23}$$

Then we have the model

$$EY = X \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + Z \begin{pmatrix} \beta_2 \\ \beta_3 \end{pmatrix} \tag{24}$$

And we can use equation (9.2) from the text

$$E\hat{\beta} = \beta + L\gamma \tag{25}$$

Where $\gamma = (\beta_2, \beta_3)'$ and

$$L\gamma = (X'X)^{-1}X'Z\gamma \tag{26}$$

Which gives us the realtion

$$E\hat{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{28} \end{pmatrix} \begin{pmatrix} 28 & 0 \\ 0 & 196 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \beta_3 \end{pmatrix} \tag{27}$$

So that the bias is

$$
\begin{aligned}
E\hat{\beta}_0 &= \beta_0 + 4\beta_2 && (28)\\
E\hat{\beta}_1 &= \beta_1 + 7\beta_3 && (29)
\end{aligned}
$$

Exercises 10a

4. *Show that* $(1 - h_i)^2 + \sum_{j \neq i} h_{ij}^2 = (1 - h_i)$.

   If the the h_{ij} are the elements of the hat matrix H which has the form

$$
\hat{Y} = X(X'X)^{-1}X'Y = HY \tag{30}
$$

   Then, H is the idempotent projection matrix that transforms Y into $\hat{Y}$. This implies $(I - H)^2 = I - H$ and shows $(1 - h_i)^2 + \sum_{j \neq i} h_{ij}^2 = (1 - h_i)$.

Exercises 10c

4. *Suppose that* $Y_1, Y_2, \ldots, Y_n$ *are gamma random variables so that* $EY = r\lambda_i = \mu_i$*, say, and* $var[Y_i] = r^{-1}\mu_i^2$*. Find a transformation that will make the variances of the* Y_i *approximately equal.*

   If we use the methods described in section 10.4.3 of the text, we find that we can make the variances more homogeneous if we use

$$
\begin{aligned}
var[f(Y_i)] &\approx \left(\frac{df}{d\mu}\right)^2 var[Y_i] && (31)\\
&= \left(\frac{df}{d\mu}\right)^2 w(\mu) && (32)
\end{aligned}
$$

   Then in this case we let $w(\mu) = r^{-1}\mu_i^2$. And we look at

$$
\begin{aligned}
f(\mu) &= \int \left(\frac{r}{\mu^2}\right)^{1/2} d\mu && (33)\\
&= \sqrt{r} \int \frac{1}{\mu} d\mu && (34)\\
&= \sqrt{r} \log \mu && (35)
\end{aligned}
$$

   So we expect the data $(\sqrt{r}\log Y_1, \sqrt{r}\log Y_2, \ldots, \sqrt{r}\log Y_n)$ to have approximately the same variance.