Homework 1

Jeremy Morris

September 5, 2006

Exercise 1.1 *Assume in the one-period binomial market of Section 1.1 that both H and T have positive probability of occurring. Show that condition (1.1.2) precludes arbitrage. In other words, show that if $X_0 = 0$ and*

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$$

then we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well, and this is the case regardless of the choice of the number Δ_0.

If $X_0 = 0$, then we get

$$X_1 = \Delta_0 \left[S_1 - (1+r)S_0\right] \tag{1}$$

Then, if the stock price goes up (which happens with positive probability), we have

$$\begin{aligned} X_1 &= \Delta_0 \left[uS_0 - (1+r)S_0\right] \\ &= S_0 \Delta_0 \left[u - (1+r)\right] \end{aligned}$$

where, from condition (1.1.2), we have $u > 1 + r$ and (1) is positive. Likewise, if the stock price goes down, we have

$$\begin{aligned} X_1 &= \Delta_0 \left[dS_0 - (1+r)S_0\right] \\ &= S_0 \Delta_0 \left[d - (1+r)\right] \end{aligned}$$

and condition (1.1.2) tells us that $d < 1 + r$ and (1) is negative. Since S_0 is always positive, the number Δ_0 only serves to switch the signs in both cases. Therefore no arbitrage holds regardless of the choice of Δ_0.

Exercise 1.4 *In the proof of Theorem 1.2.2, show under the induction hypothesis that*

$$X_{n+1}(\omega_1\omega_2 \ldots \omega_n T) = V_{n+1}(\omega_1\omega_2 \ldots \omega_n T)$$

From the text, we use the equations

$$X_{n+1}(T) = \Delta_n d S_n + (1+r)(X_n - \Delta_n S_n) \tag{2}$$

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} \tag{3}$$

Then substituting (3) into (2) and taking into account the event T, we get

$$\begin{aligned}
X_{n+1}(T) &= (1+r)X_n + (1+r)(\Delta_n dS_n - \Delta_n S_n) \\
&= (1+r)X_n + \Delta_n S_n(d-(1+r)) \\
&= (1+r)V_n + \left(\frac{V_{n+1}(H)-V_{n+1}(T)}{(u-d)S_n}\right) S_n(d-(1+r)) \\
&= (1+r)V_n + \left(\frac{d-(1+r)}{u-d}\right) V_{n+1}(H) - \left(\frac{d-(1+r)}{u-d}\right) V_{n+1}(T) \\
&= (1+r)V_n - \left(\frac{1+r-d}{u-d}\right) V_{n+1}(H) + \left(\frac{1+r-d}{u-d}\right) V_{n+1}(T) \\
&= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) - \tilde{p}V_{n+1}(H) + \tilde{p}V_{n+1}(T) \\
&= \tilde{q}V_{n+1}(T) + \tilde{p}V_{n+1}(T) \\
&= V_{n+1}(T)
\end{aligned}$$

Exercise 1.6 (Hedging a long position-one period) *Consider a bank that has a long position in the European call written on the stock price in Figure 1.1.2. The Call expires at time one and has strike price $K = 5$. In Section 1.1, we determined the time-zero price of this call to be $V_0 = 1.20$. At time zero, the bank owns this option, which ties up capital $V_0 = 1.20$. The bank wants to earn the interest rate 25% on this capital until time one (i.e., without investing any more money, and regardless of how the coin tossing turns out, the bank wants to have*

$$\frac{5}{4} \cdot 1.20 = 1.50$$

at time one, after collecting the payoff from the option (if any) at time one). Specify how the bank's trader should invest in the stock and money markets to accomplish this.

Where $S_0 = 4, S_1(H) = 8, S_1(T) = 2, r = 0.25, k = 5, X_1 = 1.50$ and $V_1 = (S_1 - k)^+$, solve

$$\begin{aligned}
X_1(H) &= \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) + V_1(H) \\
X_1(T) &= \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) + V_1(T)
\end{aligned}$$

for Δ_0 and X_0. This will tell us how much stock will need to be purchased and how much needs to be invested in the money market. Plugging in all the numbers, we get

$$\begin{aligned}
1.5 &= 8\Delta_0 + 1.25(X_0 - 4\Delta_0) + 3 \\
1.5 &= 4\Delta_0 + 1.25(X_0 - 4\Delta_0)
\end{aligned}$$

Which reduces to the system of equations

$$\begin{pmatrix} -1.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1.25 & 3 \\ 1.25 & -1 \end{pmatrix} \begin{pmatrix} X_0 \\ \Delta_0 \end{pmatrix}$$

This gives $X_0 = 0.60$ and $\Delta_0 = -0.75$. This means that we put 0.60 in a bank account and take a short position in 0.75 shares of the stock assuming that we own a European option on the stock.