Homework 2

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1 (Multiple roots of Characteristic Polynomial)

The characteristic polynomial is

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$$

with the corresponding difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = 0 \qquad (1)$$

$\therefore$

$$\rho(r) = \rho'(r) = \cdots = \rho^{(q-1)}(r) = 0$$

Now

$$\rho(r) = 0 \Rightarrow \sum_{j=0}^{k} \alpha_j r^j = 0$$

Comparing with (1) we say that $y_n = r^n$ is the solution of the difference equation.

$$\begin{aligned}
& \rho'(r) = 0 \\
\Rightarrow \quad & \sum_{j=0}^{k} j\alpha_j r^{j-1} = 0 \\
\therefore \quad & \frac{\sum_{j=0}^{k} j\alpha_j r^j}{r} = 0 \\
\therefore \quad & \sum_{j=0}^{k} j\alpha_j r^j = 0
\end{aligned}$$

Again, comparing with eqn (1), we get that $y_n = nr^n$ is a solution of the difference equation.

$$\begin{aligned} & \rho''(r) = 0 \\ \Rightarrow \quad & \sum_{j=0}^{k} j^2 \alpha_j r^{j-2} = 0 \\ \therefore \quad & \frac{\sum_{j=0}^{k} j^2 \alpha_j r^j}{r^2} = 0 \\ \therefore \quad & \sum_{j=0}^{k} j^2 \alpha_j r^j = 0 \end{aligned}$$

Again comparing with (1) we get that $y_n = n^2 r^n$ is the solution. Similarly

$$\begin{aligned} & \rho^{(q-1)}(r) = 0 \\ \Rightarrow \quad & \sum_{j=0}^{k} j^{q-1} \alpha_j r^{j-[q-1]} = 0 \\ \therefore \quad & \frac{\sum_{j=0}^{k} j^{q-1} \alpha_j r^j}{r^{q-1}} = 0 \\ \therefore \quad & \sum_{j=0}^{k} j^{q-1} \alpha_j r^j = 0 \end{aligned}$$

Comparing with (1) we get $y_n = n^{q-1} r^n$ is the solution of the difference equation.

2 (Difference Equations)

We consider the LMM

$$y_{n+2} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n)$$

applied to the IVP

$$y' = 0, \quad y(0) = 1$$

So we look at the difference equation

$$\sum_{j=0}^{k} = \alpha_j y_{n+j} = 0$$

where the solutions for the difference equation are

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j = r^2 + 4r - 5 = 0$$

So, the roots of the difference equation are $r = 1, -5$ and the general solution of the difference equation is

$$y_n = \alpha + \beta(-5)^n$$

Finding the particular solution for the starting values $y_0 = 1$ and $y_1 = 1+h$, we get

$$\begin{aligned} \alpha + \beta &= 1 \\ \alpha - 5\beta &= 1 + h \end{aligned}$$

which gives us the values $\alpha = 1 + h/6$ and $\beta = -h/6$. To show this diverges we take the limit

$$\lim_{h \to 0} (1 + \frac{h}{6}) - \frac{h}{6}(-5)^{1/h}$$

Where $n = 1/h$. Then we see

$$\lim_{n \to \infty} (1 + \frac{1}{n6}) - \frac{1}{n6}(-5)^n = \lim_{n \to \infty} -\frac{(-5)^n}{n6}$$

Since this is an indeterminate form, we use L'Hopital's rule

$$\lim_{n \to \infty} -\frac{(-5)^n \ln 5}{6}$$

This clearly diverges. If we use the starting strategy of $y_0 = 1$ and $y_1 = 1+h(-5)^{-n}$ we get a convergent method. We can look at the effect on round off error by looking at the local trucation error. For example, we see that

$$C_0 = \rho(1) = 0$$

As it should be, but

$$C_1 = \sum_{j=1}^{k} j\alpha_j = 1 + 2(1 + h(-5)^{-n}) = 3 + 2\frac{h}{(-5)^{-n}}$$

And we see that as $h \to 0$, we get that $C_1 = 3$.

3 (General explicit 3-step Method)

The order of the method is 3, $\therefore p = 3$. And

$$\begin{aligned} C_0 &= C_1 = C_2 = C_3 = 0 \\ C_4 &\neq 0 \\ C_4 &= \text{error constant of method} \end{aligned}$$

Now, the given method is

$$y_{n+3} + \alpha y_{n+2} + a y_{n+1} + b y_n = h\left[\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\right] \qquad (2)$$

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (3)$$

Comparing (2) and (3), we get

$$\begin{array}{ll} \alpha_0 = b & \beta_0 =? \\ \alpha_1 = a & \beta_1 =? \\ \alpha_2 =? & \beta_2 =? \\ \alpha_3 = 1 & \beta_3 = 0 \end{array}$$

Throughout we will be using the equations for the local trucation error. These are

$$C_0 = \sum_{j=0}^{k} \alpha_j \qquad (4)$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \qquad (5)$$

$$C_p = \frac{1}{p!} \sum_{j=1}^{k} j^p \alpha_j - \frac{1}{(p-1)!} \sum_{j=1}^{k} j^{p-1} \beta_j \qquad (6)$$

From (4) we get

$$a + b + \alpha_2 + 1 = 0$$

$$\alpha_2 = -(1 + a + b) \qquad (7)$$

From (5) we get

$$\begin{aligned} &\alpha_1 + 2\alpha_2 + 3\alpha_3 - [\beta_0 + \beta_1 + \beta_2] = 0 \\ &a + 2\alpha_2 + 3 - [\beta_0 + \beta_1 + \beta_2] = 0 \\ &\beta_0 + \beta_1 + \beta_2 = a + 2(-1 - a - b) + 3 \end{aligned}$$

And, finally

$$\beta_0 + \beta_1 + \beta_2 = -a - 2b - 1 \tag{8}$$

Then, from (6) we get

$$\begin{aligned}
C_2 &= \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j - \frac{1}{1!}\sum_{j=1}^{k} j^1\beta_j = 0 \\
&\frac{1}{2}\left[\alpha_1 + 4\alpha_2 + 9\alpha_3\right] - \{\beta_1 + 2\beta_2\} = 0 \\
&\beta_1 + 2\beta_2 = \frac{1}{2}\left[\alpha_1 + 4\alpha_2 + 9\alpha_3\right] \\
&\beta_1 + 2\beta_2 = \frac{1}{2}\left[-3a - 4b + 5\right] \qquad (9)
\end{aligned}$$

Again, from (6) we get

$$\begin{aligned}
C_3 &= \frac{1}{3!}\sum_{j=1}^{k} j^3\alpha_j - \frac{1}{2!}\sum_{j=1}^{k} j^2\beta_j = 0 \\
&\frac{1}{6}\left[\alpha_1 + 8\alpha_2 + 27\alpha_3\right] - \frac{1}{2}\left[\beta_1 + 4\beta_2\right] = 0 \\
&\frac{1}{3}\left[a + 8\alpha_2 + 27\right] = \beta_1 + 4\beta_2 \\
&\frac{1}{3}\left[-7a - 8b + 19\right] = \beta_1 + 4\beta_2 \qquad (10)
\end{aligned}$$

Solving equations (8), (9), (10), we get

$$\begin{aligned}
\beta_0 &= -1.583 + 0.0833a - 0.33b \\
\beta_1 &= -1.33 - 0.66a - 1.33b \\
\beta_2 &= 1.9166 - 0.4166a - 0.33b
\end{aligned}$$

We get the error constant by solving (6) for C_4

$$\begin{aligned}
C_4 &= \frac{1}{4!}\sum_{j=1}^{k} j^4\alpha_j - \frac{1}{3!}\sum_{j=1}^{k} j^3\beta_j \\
&= \frac{1}{4!}(\alpha_1 + 2^4\alpha_2 + 3^4\alpha_3) - \frac{1}{3!}(\beta_1 + 2^3\beta_2) \\
&= \frac{1}{24}(a + 16[-a - b - 1] + 243) - \frac{1}{6}(\beta_1 + 8\beta_2) \\
&= \frac{1}{24}(-15a - 16b + 227) - \frac{1}{6}(\beta_1 + 8\beta_2) \\
&= \frac{1}{24}(-15a - 16b + 227) - \frac{1}{6}(13.9998 - 3.9988a - 3.973b) \\
C_4 &= 0.04166a - 4.5 \times 10^{-3}b + 7
\end{aligned}$$

4 (Backward Euler Method)

For the backward euler method, the characterisitc polynomials are:

$$\rho(r) = r - 1$$

$$\sigma(r) = r$$

$$z = \rho(e^{i\theta})/\sigma(e^{i\theta}) = 1 - e^{-i\theta} = 1 - \cos\theta + i\sin\theta$$

Hence, the boundary locus is the unit circle shifted one unit to the right. This divides the complex plane into two regions: the region inside this circle and everything outside of it. All linear multistep methods are necessarily absolutely unstable for small positive values of $Re(z)$, so the region of absolute stability for the backward euler method is everything outside of the circle. This is "good" because it means that the entire left half plane is included in the region of absolute stable.

5 (Explicit Methods)

The stability polynomial is defined as

$$\pi(r,\kappa) = \rho(r) - \kappa\sigma(r)$$

We have to prove that as $\kappa \to \infty$, we cannot have any of the roots of $\pi(r,\kappa) \to \infty$ where the roots r depend on κ. We know from the Schur criterion that for a region of stability we have

$$|\alpha_0 - \kappa\beta_0| < 1$$

Which puts clear bounds on the region surrounding κ. However, it is unclear what happens when $\beta_0 = 0$. To be honest, I don't know how to procede from this point. Please don't penalize the other members of my group for my inability to solve this problem. *Jeremy Morris.*

6 (Adams Methods)

Give explicit values for the α's and β's of the Adams-Bashforth and Adams-Moulton methods.

6.1 Adams-Bashforth

Adams-Bashforth methods are defined as follows

$$y_{n+k} - y_{n+k-1} = h \sum_{j=0}^{k-1} \beta_j f_{n+j} \tag{11}$$

where the β_j are uniquely determined by the fact that $p = k$. We will also use the definition for the coefficients of the LTE. These are

$$C_0 = \sum_{j=0}^{k} \alpha_j \tag{12}$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \tag{13}$$

$$C_p = \frac{1}{p!} \sum_{j=1}^{k} j^p \alpha_j - \frac{1}{(p-1)!} \sum_{j=1}^{k} j^{p-1} \beta_j \tag{14}$$

We will find the values for the coefficients of (11) for the values $k = 1, 2, 3, 4$.

6.1.1 $k = 1$

Here the Adams method becomes

$$y_{n+1} - y_n = h\beta_0 f_n$$

Using the equations that define the LTE, we come up with the following equations

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 = 0 \\ C_1 &= \alpha_1 - (\beta_0 + \beta_1) = 0 \end{aligned}$$

From our method, we can tell that $\alpha_0 = 1, \alpha_1 = -1$ and since this method is explicit, we know that $\beta_1 = 0$. Which leaves us with the equation

$$\alpha_1 - \beta_0 = 0$$

Which tells us that $\beta_0 = -1$.

6.1.2 $k = 2$

The Adams method becomes

$$y_{n+2} - y_{n+1} = h\beta_0 f_n + h\beta_1 f_{n+1}$$

By again using the LTE equations, we get

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ C_1 &= \alpha_1 + 2\alpha_2 - (\beta_0 + \beta_1 + \beta_2) = 0 \\ C_2 &= \frac{1}{2}(\alpha_1 + 4\alpha_2) - (\beta_1 + 2\beta_2) = 0 \end{aligned}$$

This gives us the system of equations

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -1 & -1 \\ 0 & 1/2 & 2 & 0 & -1 & 2 \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For this step, we can say that $\alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 1, \beta_2 = 0$. Which leaves us with the system

$$\begin{aligned} -1 - \beta_0 - \beta_1 &= 0 \\ -1/2 - \beta_1 &= 0 \end{aligned}$$

Which gives us that $\beta_0 = 1/2, \beta_1 = -1/2$.

6.1.3 $k = 3$

Here, the Adams method becomes

$$y_{n+3} - y_{n+2} = h\beta_0 f_n + h\beta_1 f_{n+1} + h\beta_2 f_{n+2}$$

Using the LTE equations, we get

$$\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\
C_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 - (\beta_0 + \beta_1 + \beta_2 + \beta_3) = 0 \\
C_2 &= \frac{1}{2}(\alpha_1 + 4\alpha_2 + 9\alpha_3) - (\beta_1 + 2\beta_2 + 3\beta_3) = 0 \\
C_3 &= \frac{1}{6}(\alpha_1 + 8\alpha_2 + 27\alpha_3) - \frac{1}{2}(\beta_1 + 4\beta_2 + 9\beta_3) = 0
\end{aligned}$$

We know that $\alpha_0 = \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 1, \beta_3 = 0$. From the remaining conditions we get the following system of equations

$$\begin{array}{ccccccc}
\beta_0 & + & \beta_1 & + & \beta_2 & = & 1 \\
 & + & \beta_1 & + & 2\beta_2 & = & 5/2 \\
 & + & 1/2\beta_1 & + & 2\beta_2 & = & 19/6
\end{array}$$

Which tells us that $\beta_0 = 5/12, \beta_1 = -4/3, \beta_2 = 23/12$.

6.1.4 $k = 4$

Here, the Adams method becomes

$$y_{n+4} - y_{n+3} = h\beta_0 f_n + h\beta_1 f_{n+1} + h\beta_2 f_{n+2} + h\beta_3 f_{n+3}$$

Using the LTE equations, we get

$$\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\
C_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4) = 0 \\
C_2 &= \frac{1}{2}(\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4) - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4) = 0 \\
C_3 &= \frac{1}{6}(\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4) - \frac{1}{2}(\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4) = 0 \\
C_4 &= \frac{1}{24}(\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4) - \frac{1}{6}(\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4) = 0
\end{aligned}$$

We know that $\alpha_0 = \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = -1, \alpha_4 = 1, \beta_4 = 0$. From the remaining conditions we get the following system of equations

$$\begin{array}{ccccccccc}
\beta_0 & + & \beta_1 & + & \beta_2 & + & \beta_3 & = & 1 \\
 & + & \beta_1 & + & 2\beta_2 & + & 3\beta_3 & = & 7/2 \\
 & + & 1/2\beta_1 & + & 2\beta_2 & + & 9/2\beta_3 & = & 37/6 \\
 & + & 1/6\beta_1 & + & 8/6\beta_2 & + & 27/6\beta_3 & = & 175/24
\end{array}$$

And we have $\beta_0 = -3/8, \beta_1 = 37/24, \beta_2 = -59/24, \beta_3 = 55/24$.

6.2 Adams-Moulton

Adams-Moulton methods are defined as follows

$$y_{n+k} - y_{n+k-1} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{15}$$

where the β_j are uniquely determined by the fact that $p = k + 1$. We will again be using eqn (12),(13),(14) for the LTE in order to determine the coefficients.

6.2.1 $k = 1$

When $k = 1$, the Adams-Moulton method becomes

$$y_{n+1} - y_n = h\beta_0 y_n + h\beta_1 y_1$$

Using the equations defining the LTE, we get

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 \\ C_1 &= \alpha_1 - (\beta_0 + \beta_1) \\ C_2 &= \frac{1}{2}\alpha_1 - \beta_1 \end{aligned}$$

We know that for $k = 1$, $\alpha_0 = -1, \alpha_1 = 1$, which gives us the system of equations

$$\begin{array}{ccccc} \beta_0 & + & \beta_1 & = & 1 \\ & & \beta_1 & = & 1/2 \end{array}$$

So, $\beta_0 = \beta_1 = 1/2$.

6.2.2 $k = 2$

When $k = 2$, the Adams-Moulton method becomes

$$y_{n+2} - y_{n+1} = h\beta_0 y_n + h\beta_1 y_1 + h\beta_2 y_2$$

Using the equations defining the LTE, we get

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ C_1 &= \alpha_1 + 2\alpha_2 - (\beta_0 + \beta_1 + \beta_2) = 0 \\ C_2 &= \frac{1}{2}(\alpha_1 + 4\alpha_2) - (\beta_1 + 2\beta_2) = 0 \\ C_3 &= \frac{1}{6}(\alpha_1 + 8\alpha_2) - \frac{1}{2}(\beta_1 + 4\beta_2) = 0 \end{aligned}$$

We know that for $k = 2$, $\alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 1$. Which gives us the system

$$\begin{array}{rcrcrcr} \beta_0 & + & \beta_1 & + & \beta_2 & = & 1 \\ & & \beta_1 & + & 2\beta_2 & = & 3/2 \\ & & \frac{1}{2}\beta_1 & + & 2\beta_2 & = & 7/6 \end{array}$$

So, $\beta_0 = -1/12, \beta_1 = 2/3, \beta_2 = 5/12$.

6.2.3 $k = 3$

When $k = 3$, the Adams-Moulton method becomes

$$y_{n+3} - y_{n+2} = h\beta_0 y_n + h\beta_1 y_1 + h\beta_2 y_2 + h\beta_3 y_3$$

Using the equations defining the LTE, we get

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ C_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 - (\beta_0 + \beta_1 + \beta_2 + \beta_3) = 0 \\ C_2 &= \frac{1}{2}(\alpha_1 + 4\alpha_2 + 9\alpha_3) - (\beta_1 + 2\beta_2 + 3\beta_3) = 0 \\ C_3 &= \frac{1}{6}(\alpha_1 + 8\alpha_2 + 27\alpha_3) - \frac{1}{2}(\beta_1 + 4\beta_2 + 9\beta_3) = 0 \\ C_4 &= \frac{1}{24}(\alpha_1 + 16\alpha_2 + 81\alpha_3) - \frac{1}{6}(\beta_1 + 8\beta_2 + 27\beta_3) = 0 \end{aligned}$$

For $k = 3$, $\alpha_0 = \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 1$, and we have the system

$$\begin{array}{rcrcrcrcr} \beta_0 & + & \beta_1 & + & \beta_2 & + & \beta_3 & = & 1 \\ & & \beta_1 & + & 2\beta_2 & + & 3\beta_3 & = & 5/2 \\ & & 1/2\beta_1 & + & 2\beta_2 & + & 9/2\beta_3 & = & 19/6 \\ & & 1/6\beta_1 & + & 8/6\beta_2 & + & 27/6\beta_3 & = & 65/24 \end{array}$$

So, $\beta_0 = -47/12, \beta_1 = 155/16, \beta_2 = -57/8, \beta_3 = 113/48$.

6.2.4 $k = 4$

When $k = 4$, the Adams-Moulton method becomes

$$y_{n+4} - y_{n+3} = h\beta_0 y_n + h\beta_1 y_1 + h\beta_2 y_2 + h\beta_3 y_3 + h\beta_4 y_4$$

Using the equations defining the LTE, we get

$$\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\
C_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \alpha_4) = 0 \\
C_2 &= \frac{1}{2}(\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4) - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4) = 0 \\
C_3 &= \frac{1}{6}(\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4) - \frac{1}{2}(\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4) = 0 \\
C_4 &= \frac{1}{24}(\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4) - \frac{1}{6}(\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4) = 0 \\
C_5 &= \frac{1}{120}(\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4) - \frac{1}{24}(\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4) = 0
\end{aligned}$$

For $k = 4$, $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 = -1, \alpha_4 = 1$, and we have the system

$$\begin{array}{ccccccccccc}
\beta_0 & + & \beta_1 & + & \beta_2 & + & \beta_3 & + & \beta_4 & = & 1 \\
 & & \beta_1 & + & 2\beta_2 & + & 3\beta_3 & + & 4\beta_4 & = & 7/2 \\
 & & 1/2\beta_1 & + & 2\beta_2 & + & 9/2\beta_3 & + & 8\beta_4 & = & 37/6 \\
 & & 1/6\beta_1 & + & 8/6\beta_2 & + & 27/6\beta_3 & + & 64/6\beta_4 & = & 65/24 \\
 & & 1/24\beta_1 & + & 16/24\beta_2 & + & 81/24\beta_3 & + & 256/24\beta_4 & = & 781/24
\end{array}$$

So, $\beta_0 = -199/720, \beta_1 = 413/360, \beta_2 = -28/15, \beta_3 = 683/360, \beta_4 = 71/720$.

7 (Numerical Comparison)

For this particular problem, where $f(x, y) = y$, each method can be turned into an explicit method. Euler's method and the 4-th-order Runge-Kutta methods are already explicit, obviously. For the others, after plugging in the values for $f(x, y)$, we can solve for y_{n+1} in terms of y_n (or solve for y_{n+2} in terms of y_{n+1} and y_n in the case of Simpson's rule). This is what we obtain:

Euler's method $y_{n+1} = (h + 1)y_n$

Trapezoidal rule $y_{n+1} = (2 + h)/(2 - h)y_n$

Simpson's rule $y_{n+2} = (4hy_{n+1} + (h + 3)y_n)/(3 - h)$

4th-order Runge-Kutta $y_{n+1} = y_n + (h/6)(K_1 + 2K_2 + K_4)$
where $K_1 = y_n, K_2 = y_n + (h/2)K_1, K_3 = y_n + (h/2)K_2, K_4 = y_n + hK_3$

When we run each of these methods with step sizes $h = 2^{-s}$ for $s = 3, 4, \ldots, 15$, and plot the error (i.e., $e - y_n$) at the right endpoint against s we obtain the attached superimposed plots. Euler's method is shown in black, Trapezoidal rule in blue, Simpson's rule in red, and the Runge-Kutta method in yellow. It is interesting that Euler's method performs better than all the others.

8 (A complicated region of absolute stability)

See attached figure for a plot of the region of absolute stability. Notice, the region of absolute stability is not the entire left half plane. Hence the method is not unconditionally stable. The method is restricted to specific values of κ i.e pruduct of λ and h. Hence we will have to restrict ourself to small step sizes to get stability.

9 (The Reality of Absolute Stability)

Here we apply Euler's Method to the IVP

$$y' = -100(y - \sin x) + \cos x, \quad y(0) = 0, \quad x \in [0, 20]$$

The exact solution of this problem is $y(x) = \sin x$. To find the smallest value of h that will make this method unstable, we look at the stability region of Euler's method. We know that the stability region of Euler's method is defined as $e^{i\theta} - 1$, which is the unit circle shifted to the left. And we know, that for this problem, $\lambda = -100$. Since λ is real, we are only considering the real line on the interval $(-2, 0)$. Then if we solve for the left end point, we see that $h = 1/50$.

See attached image for a plot of the error $e = y(20) - y_n$ against h. Notice that for values of h smaller than specified by this method, we get very unstable results.

10 (Adaptive Quadrature)

Attached is the table generated by the MATLAB "tabulate" procedure for both problems. The safety factor was set to sigma = 0.9. The results are as to be expected: as the tolerance value τ decreases, the accuracy increases; the number of function evaluations increases; the largest and smallest step size accepted decrease.