Homework 3

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October 27, 2004

Exercise 1 : We can say that

$$\|x\| = \|x - y + y\|$$

$$\|x\| \leq \|x - y\| + \|y\|$$

$$\|x\| - \|y\| \leq \|x - y\|$$

And likewise

$$\|y\| = \|y - x + x\|$$

$$\|y\| \leq \|y - x\| + \|x\|$$

$$\|y\| - \|x\| \leq \|y - x\|$$

So we get

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\|$$

Exercise 2 : (An Induced Matrix Norm.) Let A be a real n x n matrix. Show that:

$$\|A\|_1 = \max_{j=1,2,\ldots,n} \sum_{i=1}^{n} |a_{ij}|.$$

Hint: Use the definition of the induced matrix norm:

$$\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1.$$

Answer: let

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

then

$$\|Ax\|_1 = \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}||x_j|.$$

Changing the order of summation, we can separate the summands,

$$\|Ax\|_1 \leq \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}||x_j|.$$

let

$$c = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| \tag{1}$$

then

$$\|Ax\|_1 \leq c\|x\|_1$$

and thus

$$\|A\|_1 \leq c$$

to show this as an equality, we demonstrate an x for which

$$\frac{\|Ax\|_1}{\|x\|_1} = c$$

let k be the column index for which the maximum in (1) is attained. Let $x = e_k$, the k^{th} unit vector. Then $\|x\|_1 = 1$ and

$$\|A\|_1 = \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j| = \sum_{i=1}^{n} |a_{ik}| = c$$

This proves that for the vector norm $\|\cdot\|$, the operator norm is

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$$

Exercise 3 : An induced norm is defined as

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

if we let $A = I_n$ then

$$\|A\|_F = \|I\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{1+1+\ldots+1} = \sqrt{n}$$

but

$$\|A\| = \|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1$$

Since $\max\limits_{\|x\|=1} \|Ax\| \neq \|Ax\|_F$ the Frobenius norm is not an induced norm.

Exercise 4 : Does the spectral radius itself define a norm? Why or why not? Answer. As we have seen it in class, for an arbitrary square matrix A,

$$r_\sigma(A) \leq \|A\| \tag{2}$$

Moreover, if $\varepsilon > 0$ be given, then there is an operator matrix norm, for which

$$\|A\|_\varepsilon \leq r_\sigma(A) + \varepsilon \tag{3}$$

This shows that $r_\sigma(A)$ is almost a matrix norm. But notice that it does not satisfy all the norm properties. For example $\|A\| = 0 \leftrightarrow A = 0$ but this is not neccessarly true for spectral radius of a matrix. Take this example:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix} \tag{4}$$

The spectral radius of A is 0, but A is not the zero matrix.

Exercise 5 : Derivation: Given $Ax = b$ we want to obtain an approximation $\widetilde{x}$ to x, $\tilde{x} \neq x$. This leads to

$$\|e\| = \|\tilde{x} - x\|$$

$$\frac{\|e\|}{\|x\|}(\text{ relative error})$$

Which leads to the following set of equations:

$Ax = b$	$\rightarrow$	$\|b\| \leqslant \|A\|\|x\|$	(1)
$A^{-1}b = x$	$\rightarrow$	$\|x\| \leqslant \|A^{-1}\|\|b\|$	(2)
$Ae = r$	$\rightarrow$	$\|r\| \leqslant \|A\|\|e\|$	(3)
$A^{-1}r = e$	$\rightarrow$	$\|e\| \leqslant \|A^{-1}\|\|r\|$	(4)

Dividing the smaller side of (4) by the larger side of (1), and the larger side of (4) by the smaller side of (1) gives

$$\frac{\|e\|}{\|x\|} \leqslant \|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|}$$

Similarly with (2) and (3):

$$\frac{\|r\|}{\|A^{-1}\|\|b\|} \leqslant \frac{\|A\|\|e\|}{\|x\|}$$

Which leads to

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|r\|}{\|b\|} \leqslant \frac{\|e\|}{\|x\|} \leqslant \|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|}$$

In general this inequality can be shown to be sharp trivialy by definition. That is given

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

By definition, there is some x such that

$$\|b\| = \|Ax\| = \|A\|\|x\|$$

Similarly for equations (2),(3),(4). Therefore there is some x such that for the right hand side we have

$$\frac{\|e\|}{\|A\|\|x\|} = \|A^{-1}\|\frac{\|r\|}{\|b\|} \text{ or } \frac{\|e\|}{\|x\|} = \|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|}$$

And for the left hand side we have:

$$\frac{\|r\|}{\|A^{-1}\|\|b\|} = \frac{\|A\|\|e\|}{\|x\|} \text{ or } \frac{\|r\|}{\|A^{-1}\|\|A\|\|b\|} = \frac{\|e\|}{\|x\|} \text{ (i)}$$

a) For the more specific case of $\|\cdot\|_2$:

Using SVD $\|Ax\|_2 = \|b\|$ becomes:

$$\|Ax\|_2 = \|U\Sigma V^T x\|$$

Let $x = v_1$ then

$$\|Ax\|_2^2 = \|U\Sigma V^T v_1\|_2^2 = (U\Sigma V^T v_1)^T(U\Sigma V^T v_1) = v_1^T V\Sigma^T U^T U\Sigma V^T v_1$$

Since U and V are orthogonal this becomes

$$\Sigma^T \Sigma = \|\Sigma\|_2$$

By definition of the induced matrix norm this becomes

$$\|\Sigma\|_2 = \max_{\|x\|_2=1} \|\Sigma x\|_2 or \|\Sigma\|_2 = \max_{\|x\|_2=1} \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2}$$

Since by definition $\sigma_1 = \max(\sigma_i)$, And since $\|x\|_2 = 1$.

$$\|\Sigma x\|_2 = \sigma_1 \max_{\|x\|_2=1} \sqrt{\sum_{i=1}^{n} x_i^2} = \sigma_1$$

Similarly for $\|A^{-1}r\| = \|e\|$ letting $x = u_i$ yields:

$$\|A^{-1}r\|_2 = \|A^T u_i\|_2 = \|V \Sigma^{-1} U^T\|_2 = \sigma_n^{-1} \max_{\|x\|_2=1} \sqrt{\sum_{i=1}^{n} x_i^2} = \sigma_n^{-1}$$

Similar steps for equations (2) and (3) yield

$Ax = b$	$\rightarrow$	$—\|b\| = \|Ax\|_2 = \sigma_1 \|x\|_2$	(1.a)
$A^{-1}b = x$	$\rightarrow$	$\|x\| = \|A^{-1}b\| = \sigma_n^{-1} \|b\|_2$	(2.a)
$Ae = r$	$\rightarrow$	$\|r\| = \|Ae\| = \sigma_1 \|e\|_2$	(3.a)
$A^{-1}r = e$	$\rightarrow$	$\|e\| = \|A^{-1}r\| = \sigma_n^{-1} \|r\|_2$	(4.a)

Which gives for the right hand side:

$$\frac{\|e\|}{\|x\|} = \frac{\sigma_1}{\sigma_n} \frac{\|r\|}{\|b\|}$$

and for the left hand side:

$$\frac{\|r\|}{\sigma_1 \sigma_n^{-1} \|b\|} = \frac{\|e\|}{\|x\|}$$

Therefore the inequality is sharp when A is a diagonal matrix.
b) For $\| \cdot \|_\infty$
By definition:

$$\|A\|_\infty = \max_{\|x\|=1} \|Ax\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij} x_j| \leqslant \max_i \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{kj}|$$

Supposing that the max row sum is obtained in the k^{th} row, pick x to be ± 1 based on the signs of this k^{th}-row.

$$x = \begin{pmatrix} \pm 1 \\ \pm 1 \\ \dots \\ \pm 1 \end{pmatrix}$$

Given this x the inequality (i) becomes an equality

$Ax = b$	$\rightarrow$	$—\|b\| = \max\limits_{\|x\|=1} \|Ax\|_\infty = \sum\limits_{j=1}^{n} \|a_{kj}\| = \delta$	(1.a)
$A^{-1}b = x$	$\rightarrow$	$\|x\| = \max\limits_{\|b\|=1} \|A^{-1}b\|_\infty = \sum\limits_{j=1}^{n} \|a_{kj}^{-1}\| = \rho$	(2.a)
$Ae = r$	$\rightarrow$	$\|r\| = \max\limits_{\|e\|=1} \|Ae\| = \sum\limits_{j=1}^{n} \|a_{kj}\| = \delta$	(3.a)
$A^{-1}r = e$	$\rightarrow$	$\|e\| = \max\limits_{\|r\|=1} \|A^{-1}r\|_\infty = \sum\limits_{j=1}^{n} \|a_{kj}^{-1}\| = \rho$	(4.a)

And thus, for the right hand side:

$$\frac{\|e\|}{\|x\|} = \delta\rho \frac{\|r\|}{\|b\|}$$

And for the left hand side:

$$\frac{\|r\|}{\delta\rho\|b\|} = \frac{\|e\|}{\|x\|}$$

Exercise 6 : General case A is the Hermitian (complex symmetric matrix)

i) If $A = A^*$ then

$$\langle Ax, x\rangle = \langle x, \bar{A}x\rangle = \langle x, Ax\rangle = \langle\overline{Ax, x}\rangle$$

so $\langle Ax, x\rangle$ is real

ii) If $Ax = \lambda x$, then $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle$. Since $\langle Ax, x\rangle$ and $\langle x, x\rangle$ are real λ must be real.

It follows directly that $A = A^T$ is a special case of this. Therefore, the eigenvalues of A are real.

Exercise 7 : Let λ be an eigenvalue of $A \epsilon A^{\mathrm{nxn}}$. Show that there exists an

$$i \in \{1, 2, \ldots, n\}$$

such that

$$|a_{\mathrm{ii}} - \lambda| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{\mathrm{ij}}|$$

Proof: Let x_i be the component of largest magnitude of one of the eigenvectors of A. From the ith equation of the system $(A - \lambda I)x = 0$, we have

$$(a_{\mathrm{ii}} - \lambda)x_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{\mathrm{ij}} x_j$$

Which leads to:

$$|a_{\mathrm{ii}} - \lambda| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{\mathrm{ij}}| \frac{|x_j|}{|x_i|} \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{\mathrm{ij}}|$$

Let S be a set that is the union of $k \leqslant n$ Gershgorin circles s.t. the intersection of S with all other Gershgorin circles is empty. Show that S contains precisely k eignvalues of A (counting multiplicities) Let $k = n$, this implies that A is a diagonal matrix, $A = D$, This yields n circles centered about the diagonal entries of a_{ii} with radius 0.
Give an example showing that on the other hand there may be Gershgorin circles that contain no eigenvalues of A at all. Let:

$$A = \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

The eigenvalues for this matrix are $\lambda = 1.3908\ldots, 0.3596\ldots$ which are both outside the circle of radius 0.25 centered at 1.
NOTE:
A woman by the name of Olga Taussky Todd was famous for using this theorem and worked with the flutter group to help build more efficient aircrafts during WWII. The flutter speed is extremely important to an aircrafts making and must be carefully calculated for it to be able to get off the ground.

Olga found a more simplified way to find these calculations by finding the eigenvalues and eigenvectors using the Gershgorin Theorem. "Still, matrix theory reached me only slowly," Taussky noted in a 1988 article in the American Mathematical Monthly. "Since my main subject was number theory, I did not look for matrix theory. It somehow looked for me." (IvarsP Peterson's MathTrek).

Exercise 8 : Let $A \in \mathbb{R}^{m \times l}$ and $B \in \mathbb{R}^{l \times n}$ and show that if we use partitioned matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Where $A_{ij} \in \mathbb{R}^{m_i \times l_j}$ and $B_{ij} \in \mathbb{R}^{l_i \times n_j}$. Then the matrix product can be computed using:

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

To show this, we will take the statement for the upper left block matrix AB and expand it:

$$(AB)_{ij} \quad = \quad \sum_{k=1}^{l_1} (a_{11})_{ik}(b_{11})_{kj} + \sum_{k=1}^{l_2} (a_{12})_{ik}(b_{21})_{kj} \tag{5}$$

Which can be written in terms of the original matrices A and B

$$\sum_{k=1}^{l_1} a_{ik}b_{kj} + \sum_{k=l_1+1}^{l_1+l_2} a_{ik}b_{kj} \tag{6}$$

And we can combine these sums by writing

$$\sum_{k=1}^{l_1+l_2} a_{ik}b_{kj} \tag{7}$$

In general we can write

$$(AB)_{ij} = \sum_{k=1}^{n} A_{\hat{i}k}B_{k\hat{j}} \tag{8}$$

where n is the number of row matrices of A and the number of column matrices of B. And if $A \in \mathbb{R}^{m \times l}$ with sub-matrices $A_{ij} \in \mathbb{R}^{m_i \times l_j}$ where

$l_1 + l_2 + \ldots + l_n = l$ and, $\hat{\imath}$, $\hat{\jmath}$ refer to the appropriate matrices.

Exercise 9 :
a. False. By definition, the determinant is the sum of all possible products where we pick an element from each row and column of the matrix. This means that if we take a matrix $A \in \mathbb{R}^{4\times 4}$, that we should have $\binom{4}{4} = 24$ products to sum. However, by this method we only get 8.

$$\begin{aligned}
\det(A) &= \det(A_{11})\det(A_{22}) - \det(A_{12})\det(A_{21}) \\
&= a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} \\
&\quad -a_{13}a_{24}a_{31}a_{42} + a_{13}a_{24}a_{32}a_{41} + a_{14}a_{23}a_{31}a_{42} - a_{14}a_{23}a_{32}a_{41}
\end{aligned}$$

Notice that there are only two terms with the coefficient a_{11}, if we compute the determinant using the minor expansion, we should have six such terms.

b. False, take the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

Where we have sub-matrices

$$B = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then we have rank(B) + rank(D) = 2 but rank(A) = 3. This statement could be true if rank(B) = rank(C).

Exercise 10 : To count the number of multiplications and divisions in the Cholesky decomposition, we use the equations derived in the book for finding the individual elements of the matrix L

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{jk}}{l_{jj}}$$

$$l_{ii} = \left[a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right]^{1/2}$$

The number of divisions can be seen from the first equation, there are divisions for every element below the diagonal. This is

$$\frac{n^2 - n}{2}$$

To get the number of multiplications we notice that we need the double sum

$$\sum_{i=1}^{n-1} \sum_{j=i}^{n-1} n - j$$

Using a method similar to the one used in the notes, we approximate this expression with integrals to get the number of multiplications

$$\int_1^{n-1} \int_i^{n-1} (n-j) \; \mathrm{d}j \; \mathrm{d}i = \frac{1}{6} n^3 - \frac{1}{2} n^2 + \frac{2}{3}$$

For the case when $i = j$, we use the second equation to get the sum

$$\sum_{i=1}^{n} (n-i) \approx \int_1^n (n-i) \; \mathrm{d}i = \frac{n^2}{2} - n + \frac{1}{2}$$

When we approximate using integrals, the leading term is the only one that is important. Therefore, we see that the Cholesky decomposition is of order

$$\frac{n^3}{6}$$