Homework 2

Jeremy Morris & Haimanot Kassa

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Exercise 1 : It appears that the Taylor expansion approximates the function $f(x) = e^x$ quite well. In fact the more terms you add, the better the approximation gets. This is not the case for $g(x) = \ln(x+1)$. The Taylor expansion approximates the function quite well on the interval $(0,1)$, but does not everywhere else. See figures 1 and 2 at the back of this report.

Exercise 2 : Since F is a linear function where $F : \mathbb{R}^n \to \mathbb{R}^m$, then we can write:

$$F(\vec{x}) = F(\vec{e}_1 x_1 + \vec{e}_2 x_2 + \ldots + \vec{e}_n x_n)$$

$$= x_1 F(\vec{e}_1) + x_2 F(\vec{e}_2) + \ldots + x_n F(\vec{e}_n)$$

$$= \begin{bmatrix} | & | & & | \\ F(\vec{e}_1) & F(\vec{e}_2) & \ldots & F(\vec{e}_n) \\ | & | & & | \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A\vec{x}$$

Where $\vec{e}_i \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$.

Exercise 3 : We have the following functions with their associated matrices:

$$\begin{array}{ll} F : \mathbb{R}^p \to \mathbb{R}^m & A \in \mathbb{R}^{m\times p} \\ G : \mathbb{R}^p \to \mathbb{R}^n & B \in \mathbb{R}^{p\times n} \\ H : \mathbb{R}^n \to \mathbb{R}^m & C \in \mathbb{R}^{m\times n} \end{array}$$

With the definition that $H(\vec{x}) = F(G(\vec{x}))$. And we want to show that $H(\vec{x}) = C\vec{x}$ where $C = AB$.

Proof: We need to be able to say that if $C\vec{x} = \vec{z}$, then the i^{th} component of $\vec{z}$ is :

$$z_i = \sum_{j=1}^{n} c_{ij} x_j = \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ik} b_{kj} x_j$$

We start by doing the product $G(\vec{x}) = B\vec{x} = \vec{y}$:

$$\vec{y} = \vec{b}_1 x_1 + \vec{b}_2 x_2 + \ldots + \vec{b}_n x_n$$

Where $\vec{b}_i$ is the i^{th} column of B. This gives us the following for the components of $\vec{y}$

$$y_i = \sum_{j=1}^{n} b_{ij} x_j \quad \vec{y} \in \mathbb{R}^p$$

Then we do $F(\vec{y}) = \vec{z}$ and we get

$$\vec{z} = \vec{a}_1 y_1 + \vec{a}_2 y_2 + \ldots + \vec{a}_n y_n$$

Where $\vec{a}_i$ is the i^{th} column of A. We again solve for the components of $\vec{z}$ just as we did for $\vec{y}$

$$z_i = \sum_{j=1}^{n} a_{ij} y_j \quad \vec{z} \in \mathbb{R}^m$$

Now, when we plug in for y_j we get:

$$z_i = \sum_{j=1}^{p} a_{ij} \sum_{k=1}^{n} b_{jk} x_k = \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ij} b_{jk} x_k$$

We could use up a lot of paper, but instead we'll skip a few steps and show that we can write this equation as:

$$z_i = x_1 \left(\sum_{k=1}^{p} a_{ik} b_{k1} \right) + x_2 \left(\sum_{k=1}^{p} a_{ik} b_{k2} \right) + \ldots + x_n \left(\sum_{k=1}^{p} a_{ik} b_{kn} \right)$$

Which, thankfully, we can write as:

$$z_i = \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ik} b_{kj} x_j$$

And we are done!

Exercise 4 : If a matrix $A \in \mathbb{R}^{n \times n}$ is positive definite, then we know that for every vector $\vec{x} \in \mathbb{R}^n$, where $\vec{x} \neq 0$, we get the condition that $\vec{x}^T A \vec{x} > 0$. And we have to show that every principle submatrix $A_{\{I\}}$ is also positive definite. The matrix $A_{\{I\}}$ is obtained by picking a set $I \subset \{1, 2, \ldots, n\}$ and crossing out all rows and columns whose indices are not in I.

Proof: We are given that A is positive definite for all vectors $\vec{x} \in (R)^n$. Then we can define the vector

$$\vec{x'} = \sum_{i \in I} \alpha_i e_i$$

For scalar constants α_i and e_i are the unit vectors in $\mathbb{R}^n$. Then we can say that for the principle submatrix $A_{\{I\}}$ that

$$\vec{y}^T A_{\{I\}} \vec{y} = \vec{x'}^T A \vec{x'} > 0$$

For all vectors $\vec{y} \in \mathbb{R}^k$. And all principal submatrices of A are positive definite if A is positive definite.

Exercise 5 : To compute the factorization $A = UL$ where U is upper triangular with 1s along the diagonal and L is lower triangular, we define the matrices U and L in the following way:

```
for i = (n − 1) . . . 1
    for k = (i + 1) · · · 1
        U_ik = a_ik/a_kk
        for j = 1 · · · k
            L_ij = a_ij − a_kj U_ik
```

Adding the requirement that $U_{ii} = 1$ (since the algorithm above does not define them). Then, if we are trying to solve the system $A\vec{x} = \vec{b}$, we apply backward substitution to $\vec{b}$ by doing the following calculations:

```
for i = (n − 1) · · · 1
    for j = 1 · · · i
        b_i = b_i − b_j u_ij
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Then we use forward substitution with L to find $\vec{x}$. This is done in the following way:

$$x_i = \frac{b_i - \sum_{j=k+1}^{n} x_j l_{ij}}{l_{ii}} \text{ for } i = n, n-1, \ldots, 1$$

Exercise 6 :
part a. We say that the function f has a root α of multiplicity $p > 1$ if

$$f(x) = h(x)(x-\alpha)^p \tag{1}$$

Where $h(\alpha) \neq 0$ and $h(x)$ is continuous at $x = \alpha$. If $h(x)$ is sufficiently differentiable at $x = \alpha$, then we have

$$f(\alpha) = f'(\alpha) = \ldots = f^{(p)}(\alpha) = 0, \quad f^{(p)}(\alpha) \neq 0$$

Newton's method is defined as

$$x_{n+1} = g(x_n) \quad g(x) = x - \frac{f(x)}{f'(x)} \quad x \neq \alpha \tag{2}$$

To show that Newton's method converges linearly, we need $|g'(\alpha)| < 1$. Before calculating $g'(\alpha)$, we use 1 to get

$$g(x) = x - \frac{(x-\alpha)h(x)}{ph(x) + (x-\alpha)h'(x)}$$

Taking the derivative, we get

$$\begin{aligned} g'(x) = \quad 1 \quad &- \quad \frac{((x-\alpha)h'(x) + h(x))(ph(x) + (x-\alpha)h'(x))}{(ph(x) + (x-\alpha)h'(x))^2} \\ &- \quad \frac{(ph'(x) + (x-\alpha)h''(x) + h'(x))(h(x)(x-\alpha))}{(ph(x) + (x-\alpha)h'(x))^2} \end{aligned}$$

And for $g'(\alpha)$ we get

$$g'(\alpha) = 1 - \frac{ph(\alpha)}{p^2h(\alpha)^2} = 1 - \frac{1}{p}$$

Since $p > 1$ we get that Newton's method converges linearly with a rate of convergence of $(p-1)/p$.
part b. To show that the modification of Newton's method

$$x_{n+1} = g(x_n) \quad g(x) = x - p\frac{f(x)}{f'(x)} \quad x \neq \alpha \tag{3}$$

converges quadratically, we need to show that $|g'(\alpha)| = 0$. Following the same method as before, we use 1 to find $g'(x)$ and then we plug in α.

$$g(x) = x - p\frac{(x-\alpha)h(x)}{ph(x) + (x-\alpha)h'(x)}$$

Then we get the derivative

$$g'(x) = 1 - \frac{(ph(x) + ph'(x)(x-\alpha))(ph(x) + h'(x)(x-\alpha))}{(ph(x) + (x-\alpha)h'(x))^2}$$
$$- \frac{(ph'(x) + h'(x) + (x-\alpha)h''(x))(p(x-\alpha)h(x))}{(ph(x) + (x-\alpha)h'(x))^2}$$

When we plug in α, we get

$$g'(\alpha) = 1 - \frac{(ph(x))^2}{(ph(x))^2} = 1 - 1 = 0$$

And we have proven that the new form 3 of Newton's method converges quadratically.

Exercise 7 : To find a linear function $l(x) = ax + b$ such that

$$F = \int_0^1 (e^x - l(x))^2 \, \mathrm{d}x = min$$

Then we take the partial derivatives and set them to zero:

$$\frac{\partial F}{\partial a} = 0 \text{ and } \frac{\partial F}{\partial b} = 0$$

$$\frac{\partial F}{\partial a} = \int_0^1 \frac{\partial}{\partial a}(e^x - ax - b)^2 \, \mathrm{d}x =$$
$$\frac{-a}{3} - \frac{-b}{2} + 1 = 0$$

Solving for the partial of b we get:

$$\frac{\partial F}{\partial b} = \int_0^1 \frac{\partial}{\partial b}(e^x - ax - b)^2 \, \mathrm{d}x =$$

$$e - \frac{a}{2} - b - 1 = 0$$

And solving the system of equations for a and b:

$$\begin{cases} \frac{a}{3} + \frac{b}{2} &= 1 \\ \frac{a}{2} + b &= e - 1 \end{cases}$$

We get

$$a = \frac{6 - 3(4e - 10)}{2}$$
$$b = 4e - 10$$

Then $l(x) = ax + b$ satisfies

$$F = \int_0^1 (e^x - l(x))^2 \, \mathrm{d}x = min$$

Exercise 8 : Find a linear function $l(x) = ax + b$ such that

$$\int_0^1 |e^x - l(x)| \, \mathrm{d}x = \min$$

What we need to do in this case is minimize the area between the functions $l(x)$ and e^x. So we define the function F as

$$F(\alpha, \beta, a, b) = \int_0^\alpha e^x - ax - b \, \mathrm{d}x + \int_\alpha^\beta ax + b - e^x \, \mathrm{d}x + \int_\beta^1 e^x - ax - b \, \mathrm{d}x$$

Then we need to set $\bigtriangledown F = 0$, and we get these equations

$$\frac{\partial F}{\partial \alpha} = 2e^\alpha - 2a\alpha - b = 0$$
$$\frac{\partial F}{\partial \beta} = a\beta + 2b - 2e^\beta = 0$$
$$\frac{\partial F}{\partial a} = \alpha^2 + \beta^2 - \frac{1}{2} = 0$$
$$\frac{\partial F}{\partial b} = -2\alpha + 2\beta - 1 = 0$$

Now, we can solve for the constants α, β, a, b. Since the values α and β are the x values where $l(x)$ and e^x have the same y values, we can solve for α and β using the partial derivatives for a and b, then use the point/slope formula to find a and b. Using this method we get

$$\alpha = \frac{\sqrt{5} - 1}{4}$$
$$\beta = \frac{\sqrt{5} + 2}{4}$$

If $f(x) = e^x$, then we have the two points $(\alpha, f(\alpha)), (\beta, f(\beta))$. Solving for a and b, we get

$$a = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} = \frac{e^{2+\sqrt{5}/4} - e^{\sqrt{5}-1/4}}{\frac{2+\sqrt{5}}{4} - \frac{\sqrt{5}-1}{4}}$$

$$b = -a\left(\frac{\sqrt{5}-1}{4}\right) + e^{\frac{\sqrt{5}-1}{4}}$$

Then the function $l(x) = ax + b$ minimizes

$$\int_0^1 |e^x - l(x)| \, \mathrm{d}x$$

Exercise 9 : Find a function $l(x) = ax + b$ so that

$$f(x) = \max_{0 \leq x \leq 1} |e^x - l(x)| = min$$

What we need to do is minimize the maximum distance between $l(x)$ and e^x, the maximums occur at the points $x = 0, x = 1, x = \alpha$, see figure 3 for an explination of where these points are. Using the points $x = 0$ and $x = 1$, we can get the value of a. We require that $g(0) = g(1)$, where $g(x)$ is the distance between the functions e^x and $l(x)$. For $x = 0$ we get

$$|e^0 - a(0) - b| = |1 - b|$$

Setting this equal to the equation for $x = 1$, we have

$$|e^1 - a(1) - b| = |1 - b|$$

$$a = e - 1$$

To find α, we need to translate $y = ax + b$ so that it becomes the tangent line to e^x.

$$y' = e^x = a = e - 1$$

$$x = \ln e - 1 = \alpha$$

Thus to find b, we have

$$|e^\alpha - a\alpha - b| = |1 - b|$$

And take $-(1-b)$ to get

$$b = \frac{e^{\alpha} - a\alpha + 1}{2}$$

And our function $l(x) = ax + b$ minimizes $f(x)$.

Exercise 10 : Meg's problem here is that, for a general $n \times n$ system, we will always have to inspect all n^2 elements of the system.

Exercise 11 : The function $f(x) = x(x+1)(x-1)$ will cycle in Newton's method. That is $x_{n+2} = x_n$, given we choose the correct starting value x_0.

Exercise 12 : To begin, we consider the root finding problem

$$f(x) \equiv a - \frac{1}{x} = 0$$

For a given $a > 0$. Then we want to find the root of the left hand side, call this α. We let $a = 1/x$ be an approximate solution of this equation. At our first approximation, $(x_0, f(x_0))$, we draw the tangent line to the graph $y = f(x)$ and let x_1 be the point where this line intersects the x-axis. This will be a better approximation to α. To find the equation for x_1 we match the slope of the line that goes through the two points $(x_0, f(x_0))$ to $(x_1, 0)$ and $f'(x_0)$. We get the equation

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$

Since we know $f(x_0)$ and $f'(x_0)$ we can write

$$\frac{1}{x_0^2} = \frac{a - 1/x_0}{x_0 - x_1}$$

$$x_0 - x_1 = x_0^2(a - 1/x_0)$$

$$x_1 = x_0(2 - ax_0)$$

Then we can generalize this to get the iterative form

$$x_{n+1} = x_n(2 - ax_n)$$

If we want to know the rate of convergence and be able to determine how sensitive to the choice of x_0 this method is, then we define the residual to be $r_n = 1 - ax_n$ so that we get

$$x_{n+1} = x_n(1 + r_n)$$

And then the error can be defined as

$$e_n = \frac{1}{a} - x_n = \frac{r_n}{a}$$

To better analyze the rate of convergence, we notice that

$$r_{n+1} = 1 - ax_{n+1} = 1 - ax_n(1 + r_n) = 1 - (1 - r_n)(1 + r_n) = r_n^2$$

And we see that e_{n+1} can be defined as

$$e_{n+1} = \frac{r_{n+1}}{a} = \frac{r_n^2}{a} = \frac{e_n^2 a^2}{a} = ae_n^2$$

Based on this, we can say that this method converges quadratically.

Exercise 13 : **part a** Show that $A^{-1} = [x_{ij}]$ where

$$x_{ij} = \begin{cases} \dfrac{(n+1-j)i}{n+1} & i = 1, 2, \dots, j \\ \dfrac{(n+1-i)j}{n+1} & i = j+1, \dots, n \end{cases}$$

And

$$A_{ij} = \begin{cases} 2, & i = j \\ -1 & i = j-1, i = j+1 \\ 0 & otherwise \end{cases}$$

We will show that $AA^{-1} = I$. If we let

$$I_{ij} = c_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj}$$

We know, from the definition of A that we will get zeros in this sum, except for three terms, so the sum becomes

$$I_{ij} = c_{ij} = \sum_{k=i-1}^{i+1} a_{ik} x_{kj}$$

Then we have five cases to consider

Case 1 $(i = j)$

$$c_{ii} = (-1)\left(\frac{(n+1-i)(i-1)}{n+1}\right) + (2)\left(\frac{(n+1-i)i}{n+1}\right) + (-1)\left(\frac{(n+1-(i+1))i}{n+1}\right)$$

Then, to verify that this is 1, we expand to

$$c_{ii} = \frac{-ni - i + i^2 + n + 1 - i + 2ni + 2i - 2i^2 - ni + i^2}{n+1}$$

$$\Rightarrow \frac{n+1}{n+1} = 1$$

Case 2 $(i = j + 1)$

$$c_{ij} = (-1)\left(\frac{(n+1-j)(i-1)}{n+1}\right) + (2)\left(\frac{(n+1-i)j}{n+1}\right) + (-1)\left(\frac{(n+1-(i+1))j}{n+1}\right)$$

To verify, we substitute $i = j + 1$ and expand

$$c_{ij} = \frac{-nj - j + j^2 + 2nj - 2j^2 = nj + j^2 + j}{n+1} = 0$$

Case 3 $(i = j - 1)$

$$c_{ij} = (-1)\left(\frac{(n+1-j)(i-1)}{n+1}\right) + (2)\left(\frac{(n+1-j)i}{n+1}\right) + (-1)\left(\frac{(n+1-j)(i+1)}{n+1}\right)$$

To verify, we substitute $i = j - 1$ and expand

$$c_{ij} = \frac{-nj - j + j^2 + 2n + 2 - 2j + 2nj + 2j - 2j^2 - 2n - 2 + 2j - nj - j + j^2}{n+1} = 0$$

Case 4 $(i > j + 1)$

$$c_{ij} = (-1)\left(\frac{(n+1-(i-1))j}{n+1}\right) + (2)\left(\frac{(n+1-i)j}{n+1}\right) + (-1)\left(\frac{(n+1-(i-1))j}{n+1}\right)$$

Then we expand this substituting $i = j + c$ where $c > 1$

$$c_{ij} = \frac{j(2(n-j-c+1) - (n-j+2) - (n-j-c+2)}{n+1} = 0$$

Case 5 $(i < j - 1)$

$$c_{ij} = (-1)\left(\frac{(n+1-j)(i-1)}{n+1}\right) + (2)\left(\frac{(n+1-j)i}{n+1}\right) + (-1)\left(\frac{(n+1-j)(i+1)}{n+1}\right) = 0$$

Then we expand this substituting $i = j - c$ where $c > 1$

$$c_{ij} = \frac{(n+1-j)(2(j-c) - (j-c-1) - (j-c+1))}{n+1} = 0$$

This completes the proof.
part b We want to show that $A = LU$ for the given definitions of L and U. We make the note that to get A_{ij} we do the following

$$a_{ij} = \sum_{k=1}^{n} l_{ik}u_{kj} = l_{i(i-1)}u_{(i-1)i} + l_{ii}u_{ii}$$

Case 1 $(i = j)$

$$a_{ij} = \frac{-(i-1)}{i}(-1) + \frac{i+1}{i}(1) = \frac{2i}{i} = 2$$

Case 2 $(i = j+1)$

$$a_{ij} = \frac{-(i-1)}{i}\frac{(i+1)}{i} = \frac{-(j)}{j+1}\frac{(j+1}{j} = -1$$

Case 3 $(i = j-1)$

$$\text{I can't seem to figure this one out, but it should } = -1$$

Case 4 $(i > j+1)$ This is trivially 0
Case 3 $(i < j-1)$ This is also trivially 0
This completes the proof.
part c The advantage of using either approach is that we do not have to store any of those matrices. I would prefer to use the LU factorization, because computing the elements of the inverse seems a lot more computationally complex. We would be repeating a lot of calculations, making the inverse the least desireable option. When using the LU factorization we are only required to compute the elements of L and U once during the backwards and forwards substitutions.

Exercise 14 : Let's consider the function $f(\vec{x}) = A\vec{x} - I\vec{b}$. Then applying Newton's method, we get the iterative formula

$$\vec{x}_{n+1} = \vec{x}_n - \frac{f(\vec{x_n})}{f'(\vec{x}_n)}$$

to get a sequence of vectors. If we would like to say that this sequence of vectors is converging to something, then we will have to specify some criteria for convergence. In other words, for Newton's method to work, we will have to take some norm of the vectors $\vec{x}_n$ to determine if they are converging. We would pick the norm that would be appropriate to our problem.

As usual, we would have to pick an initial approximation that is close to the intended solution. If our matrix A does not have full rank, Newton's method may or may not converge.

Exercise 15 : Given $|\alpha - x_{n+1}| \le c_n|\alpha - x_n|$ and $c_n \to 0$ as $n \to \infty$, show

$$\lim_{n\to\infty}\left|\frac{\alpha - x_n}{x_{n+1} - x_n}\right| = 1$$

Proof : We will to show the following

$$1 \le \lim_{n\to\infty}\left|\frac{\alpha - x_n}{x_{n+1} - x_n}\right| \le 1$$

The right hand side proceeds as

$$|\alpha - x_n| \le |\alpha - x_{n+1}| + |x_{n+1} - x_n|$$

$$|\alpha - x_n| - |\alpha - x_{n+1}| \le |x_{n+1} - x_n|$$

$$\left|\frac{\alpha - x_n}{x_{n+1} - x_n}\right| - \left|\frac{\alpha - x_{n+1}}{x_{n+1} - x_n}\right| \le 1$$

$$\left|\frac{\alpha - x_n}{x_{n+1} - x_n}\right| - c_n\left|\frac{\alpha - x_n}{x_{n+1} - x_n}\right| \le 1 \tag{4}$$

$$\frac{|\alpha - x_n|}{|x_{n+1} - x_n|} \le \frac{1}{1 - c_n} = 1 \tag{5}$$

Since $c_n \to 0$ as $n \to \infty$.
Then for the left hand side we have

$$|x_{n+1} - x_n| \le |\alpha - x_n| + |\alpha - x_{n+1}|$$

$$|x_{n+1} - x_n| - \underbrace{|\alpha - x_n|}_{\le c_n|\alpha - x_n|} \le |\alpha - x_{n+1}|$$

$$|x_{n+1} - x_n| \le |\alpha - x_n|(1 + c_n)$$

$$|\alpha - x_n|(1 + c_n) \ge |x_{n+1} - x_n|$$

$$\frac{|\alpha - x_n}{x_{n+1} - x_n} \geq \frac{1}{1 + c_n} = 1$$

Since $c_n \to 0$ as $n \to \infty$.

Exercise 16 : We will show that the algorithm *Aitken* converges quadratically. To do this we show that the equation

$$G(x) = x + \frac{(g(x) - x)^2}{g(x) - x - g(g(x)) + g(x)} \tag{6}$$

has the following properties, namely that $G(\alpha) = \alpha$ and $G'(\alpha) = 0$. We define the function $g(x) = \alpha + (x - \alpha)h(x)$ for some function $h(x)$ bounded about $x = \alpha$. And we know that $g(\alpha) = \alpha$ and $g'(\alpha) \neq 0$. Then to show that $G(\alpha) = \alpha$, we find

$$G(\alpha) = \alpha + \frac{(g - \alpha)^2}{g - \alpha - (g(g) - g)}$$

Where $g = g(x)$. Notice that this will give us an indeterminant form, since $g(\alpha) = \alpha$ we have

$$G(\alpha) = \alpha + \frac{(\alpha - \alpha)^2}{\alpha - \alpha - (\alpha - \alpha)} = \alpha + \frac{0}{0}$$

So, we apply L'Hopital's rule

$$G(x) = x + \frac{-2(g - x)}{2g' - 1 - g'(g)g'}$$

And plug in for α

$$G(\alpha) = \alpha + \frac{-2(g - \alpha)}{2g' - 1 - g'(g)g'}$$

Notice that the denominator is something nonzero, in which case we have

$$G(\alpha) = \alpha + \frac{0}{2g' - 1 - g'(g)g' \neq 0} = \alpha$$

To show that $G'(\alpha) = 0$, we substitute $g(x)$ and get

$$G(x) = x + \frac{((x - \alpha)(h(x) - 1))^2}{(x - \alpha)(h(x)h(g) - 1)}$$

To make the notation a little simpler, we will specify that

$$\delta(x) = ((x - \alpha)(h(x) - 1))^2$$

$$\varphi(x) = (x-\alpha)(h(x)h(g)-1)$$

With derivatives

$$\delta'(x) = 2((x-\alpha)(h(x)-1))\left[h'(x)(x-\alpha) + (h(x)-1)\right]$$

$$\varphi'(x) = (h(x)h(g)-1) + (x-\alpha)(h'(x)h(g) + h(x)h'(g)g')$$

Then we have that the derivative of $G(x)$ is

$$G'(x) = 1 + \frac{\delta'\varphi - \delta\varphi'}{(\varphi)^2}$$

Notice that $\delta(\alpha) = \delta'(\alpha) = \varphi(\alpha) = 0$ this is an indeterminant form at $G'(\alpha)$. So we will again use L'Hopital's rule to get

$$G'(x) = 1 + \frac{\delta'\varphi' + \delta''\varphi - (\delta'\varphi' + \delta\varphi'')}{2(\varphi)\varphi'} = 1 + \frac{\delta''\varphi - \delta\varphi''}{2\varphi\varphi'}$$

From the previous calculations, we know that $\delta(\alpha) = 0$ and $\varphi(\alpha) = 0$, giving us another indeterminant form, so we use L'Hopital again

$$G'(x) = 1 + \frac{\delta''\varphi' - \delta'\varphi''}{2(\varphi')^2} \tag{7}$$

We would like to find $\delta''(\alpha)$ without having to go through all the work. So we notice the following

$$\delta'' = \overbrace{2(x-\alpha)^2}^{a}\overbrace{(h(x)-1)}^{b}\overbrace{(h'(x)(x-\alpha) + (h(x)-1))}^{c}$$

And note that

$$\delta'' = a'bc + ab'c + abc'$$

Since we will eventually be evaluating at α, the second two terms are zero(because $a(\alpha) = 0$). So we only need to find the first term. We find this, and plug into 7, evaluating at α.

$$G'(\alpha) = 1 + \frac{2(h(\alpha)-1)^2 - (2h(\alpha) - h(\alpha)^2 - 1)}{2(2h(\alpha) - h(\alpha)^2 - 1)^2}$$

Which then gives us

$$G'(\alpha) = 1 + \frac{(h(\alpha)-1)^2}{2h(\alpha) - h(\alpha)^2 - 1} = 1 - 1 = 0$$

And we see that this algorithm converges quadratically.

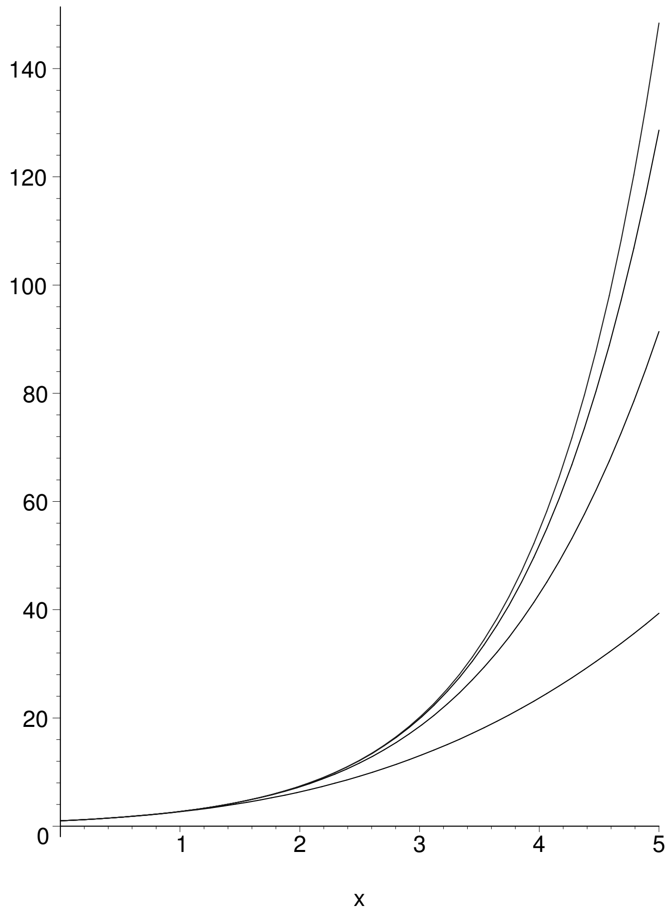

Figure 1: e^x and it's Taylor approximation

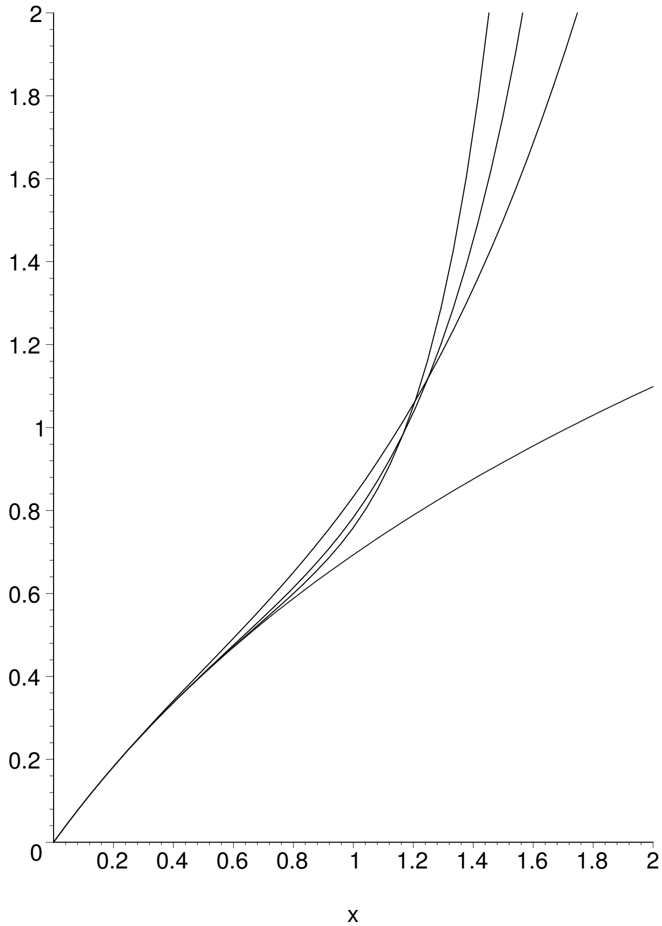

Figure 2: $\ln x + 1$ and it's Taylor approximation

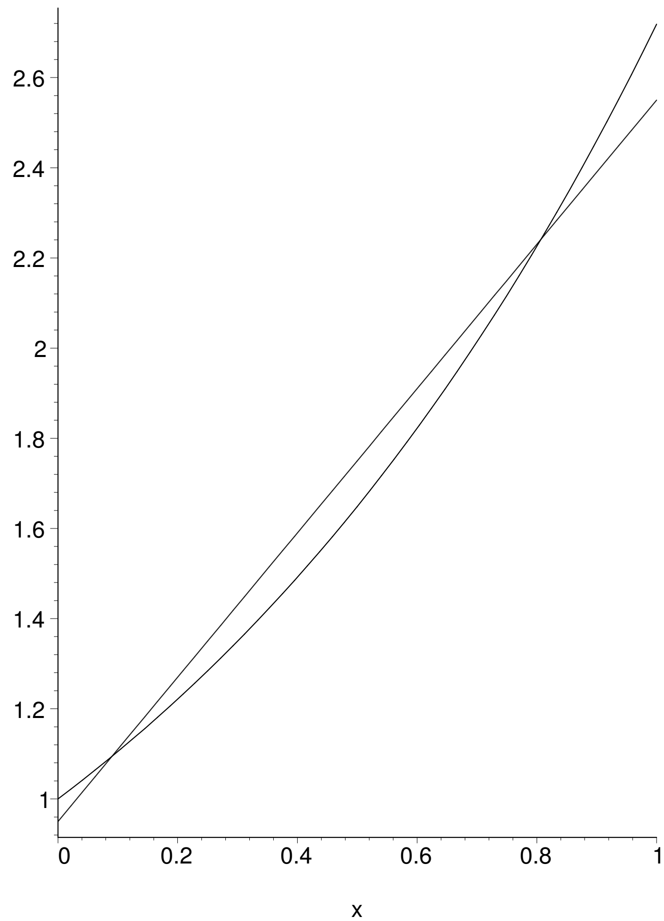

Figure 3: Plots for e^x and $l(x)$, problem 9