Introduction to Numerical Analysis I
Handout 10

1 Numerical Differentiation and Integration (cont)

1.2 Numerical Integration

1.2.1 Integration using Interpolation

Rectangular (a.k.a Endpoint) Rule Using zero polynomial (constant) interpolation at point a

$$\int_a^{a+h} f(x)\,dx \approx \int_a^{a+h} f(a) + \underbrace{f[a,x](x-a)}_{\mathcal{O}(h)}\,dx = [f(a)\,x]_a^{a+h} + \int_a^{a+h} f[a,x](x-a)\,dx \underset{\text{MVT for Integrals}}{\overset{}{\underset{x-a>0}{=}}}$$

$$f(a)\,h + f[a,\tilde{c}]\int_a^{a+h}(x-a)\,dx = f(a)\,h + f'(c)\left.\frac{(x-a)^2}{2}\right|_a^{a+h} = f(a)\,h + \underbrace{f'(c)\frac{h^2}{2}}_{error=\mathcal{O}(h^2)}$$

Proposition 1.1. $f[x_0,...,x_n,x] = f[x_0,...,x_{n+1},x](x-x_{n+1}) + f[x_0,...,x_{n+1}]$
Proof:

$$f[x_{n+1},x_0,...,x_n,x] = \frac{f[x_0,...,x_n,x] - f[x_{n+1},x_0,...,x_n]}{x-x_{n+1}}$$

Midpoint Rule Using zero polynomial (constant) interpolation at midpoint $a+\frac{h}{2}$

$$\int_a^{a+h} f(x)\,dx \approx \int_a^{a+h} f\left(a+\frac{h}{2}\right) + \underbrace{f\left[a+\frac{h}{2},x\right]\left(x-a-\frac{h}{2}\right)}_{\mathcal{O}(h)}\,dx = \left[f\left(a+\frac{h}{2}\right)x\right]_a^{a+h} + \int_a^{a+h} f\left[a+\frac{h}{2},x\right]\left(x-a-\frac{h}{2}\right)dx =$$

$$f\left(a+\frac{h}{2}\right)h + \int_a^{a+h} f\left[a+\frac{h}{2},a,x\right](x-a)\left(x-a-\frac{h}{2}\right)dx + \cancel{\int_a^{a+h} f\left[a+\frac{h}{2},a\right]\left(x-a-\frac{h}{2}\right)dx} =$$

$$f\left(a+\frac{h}{2}\right)h + \frac{f''(c)}{2}\int_a^{a+h}\left(x-a-\frac{h}{2}\right)(x-a)\,dx = f\left(a+\frac{h}{2}\right)h + \frac{f''(c)}{2}\int_0^h\left(y-\frac{h}{2}\right)y\;dy =$$

$$f\left(a+\frac{h}{2}\right)h + \frac{f''(c)}{2}\left(\frac{y^3}{3}-\frac{hy^2}{4}\right)_0^h = f\left(a+\frac{h}{2}\right)h + \frac{f''(c)}{2}\left(\frac{h^3}{3}-\frac{h^3}{4}\right) = f\left(a+\frac{h}{2}\right)h + \frac{f''(c)}{2}\frac{1}{12}h^3 =$$

$$f\left(a+\frac{h}{2}\right)h + \underbrace{\frac{f''(c)}{24}h^3}_{error=\mathcal{O}(h^3)}$$

Trapezoidal Rule Using linear interpolation at points $a, a+h$

$$\int_a^{a+h} f(x)\,dx \approx \int_a^{a+h} f(a)\frac{x-(a+h)}{-h} + f(a+h)\frac{x-a}{h} + \underbrace{\frac{f''(c)}{2}(x-a)(x-a-h)}_{\mathcal{O}(h^2)}\,dx =$$

$$= \left[f(a)\left(-\frac{x^2}{2h}+\frac{(a+h)x}{h}\right) + f(a+h)\left(\frac{x^2}{2h}-\frac{ax}{h}\right)\right]_a^{a+h} + \frac{f''(c)}{2}\int_0^h s(s-h)\,dx =$$

$$= h\frac{f(a)+f(a+h)}{2} + \frac{f''(c)}{2}\left(\frac{s^3}{3}-\frac{hs^2}{2}\right)_0^h = h\frac{f(a)+f(a+h)}{2} - \underbrace{\frac{f''(c)}{12}h^3}_{error=\mathcal{O}(h^3)}$$

Simpson's Rule Using quadratic interpolation at points $a-h, a, a+h$

$$\int_{a-h}^{a+h} f(x)\,dx \approx \int_{a-h}^{a+h} f(a-h) + f[a-h,a](x-a+h) + f[a-h,a,a+h](x-a+h)(x-a)\,dx +$$

$$\int_{a-h}^{a+h} f[a-h,a,a+h,x](x-a+h)(x-a)(x-a-h)\,dx = 2hf(a-h) + f[a-h,a]\left.\frac{(x-a+h)^2}{2}\right|_{a-h}^{a+h} +$$

$$f[a-h,a,a+h]\int_{-h}^{h}(y+h)y\,dy + \cancel{\int_{-h}^{h} f[a,a-h,a,a+h](y+h)y(y-h)\,dy} + \int_{-h}^{h} f[a,a-h,a,a+h,x](y+h)y^2(y-h)\,dy =$$

$$= 2hf(a-h) + 2h^2 f[a-h,a] + \frac{f[a,a+h]-f[a-h,a]}{2h}\left(\frac{y^3}{3}+h\frac{y^2}{2}\right)_{-h}^{h} + \frac{f^{(4)}(c)}{24}\int_{-h}^{h}(s+h)s^2(s-h)\,dx =$$

$$= 2hf(a-h) + 2h^2\frac{f(a)-f(a-h)}{h} + \frac{2h^3}{3}\frac{(f(a+h)-f(a))-(f(a)-f(a-h))}{2h^2} - \frac{f^{(4)}(c)}{24}\frac{4h^5}{15} =$$

$$= \frac{h}{3}\left[f(a-h)+4f(a)+f(a+h)\right] - \underbrace{\frac{f^{(4)}(c)}{90}h^5}_{error=\mathcal{O}(h^5)}$$

Note: One derives Simpson's rule as a linear combination of the Trapezoidal rule and the Midpoint rule.

$$\int_{a-h}^{a+h} f(x)\,dx = \frac{1}{3}\left\{\underbrace{((a+h)-(a-h))\frac{f(a-h)+f(a+h)}{2}}_{\text{Trapezoid}} + 2\underbrace{((a+h)-(a-h))f(a)}_{\text{Midpoint}}\right\}$$

Fixed Trapezoidal Rule Using Hermit interpolation interpolation at points $a, a+h$

$$\int_{a}^{a+h} f(x)\,dx \approx \int_{a}^{a+h} f(a) + f[a,a](x-a) + f[a,a,a+h](x-a)^2 + f[a,a,a+h,a+h](x-a)^2(x-a-h)\,dx +$$

$$\int_{a}^{a+h} f[a,a,a+h,a+h,x](x-a)^2(x-a-h)^2 dx = hf(a) + f'(a)\frac{h^2}{2} + \frac{f[a,a+h]-f'(a)}{h}\frac{h^3}{3} - \frac{f'(a+h)-2f[a,a+h]+f'(a)}{h^2}\frac{h^4}{12}$$

$$+\frac{f^{(4)}(c)}{24}\int_0^h y^2(y-h)^2 dx = hf(a) + f'(a)\frac{h^2}{6} + f[a,a+h]\frac{h^2}{2} - \frac{f'(a+h)+f'(a)}{h^2}\frac{h^4}{12} + \frac{f^{(4)}(c)}{24}\frac{h^5}{30} = hf(a) + f'(a)\frac{h^2}{6}$$

$$+\frac{f(a+h)-f(a)}{h}\frac{h^2}{2} - \frac{f'(a+h)+f'(a)}{h^2}\frac{h^4}{12} + \frac{f^{(4)}(c)}{24}\frac{h^5}{30} = (f(a)+f(a+h))\frac{h}{2} + (f'(a)-f'(a+h))\frac{h^2}{12} + \underbrace{\frac{f^{(4)}(c)}{720}h^5}_{error=\mathcal{O}(h^5)}$$

Example 1.2.

Exact Integration: $\int_{\pi/4}^{\pi/2}\sin x\,dx = -\cos x|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{\sqrt{2}} \approx 0.707$

Rectangular Rule: $\int_{\pi/4}^{\pi/2}\sin x\,dx \approx \int_{\pi/4}^{\pi/2}\sin\frac{\pi}{4}\,dx = \frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\pi}{4}\frac{1}{\sqrt{2}} \approx 0.55$

Midpoint Rule: $\int_{\pi/4}^{\pi/2}\sin x\,dx = \int_{\pi/4}^{\pi/2}\sin\frac{3\pi}{8}\,dx \approx \frac{\pi}{4}\cdot\sin\frac{3\pi}{8} \approx 0.726$

Trapezoidal Rule: $\int_{\pi/4}^{\pi/2}\sin x\,dx \int_{\pi/4}^{\pi/2}\sin\frac{3\pi}{8}\,dx \approx \frac{\pi}{4}\frac{\sin\frac{\pi}{4}+\sin\frac{\pi}{2}}{2} = \frac{\pi}{4}\frac{\frac{1}{\sqrt{2}}+1}{2} \approx 0.67$

Simpson Rule: $\int_{\pi/4}^{\pi/2}\sin x\,dx \int_{\pi/4}^{\pi/2}\sin\frac{3\pi}{8}\,dx \approx \frac{1}{3}\frac{\pi}{8}\left[\sin\frac{\pi}{4} + 4\sin\left(\frac{3\pi}{8}\right) + \sin\frac{\pi}{2}\right] \approx 0.7072$

Fixed Trapezoid: $\int_{\pi/4}^{\pi/2}\sin x\,dx \int_{\pi/4}^{\pi/2}\sin\frac{3\pi}{8}\,dx \approx \frac{1}{2}\frac{\pi}{4}\left[\sin\frac{\pi}{4} + \sin\frac{\pi}{2}\right] + \frac{1}{12}\frac{\pi^2}{16}\cos\frac{\pi}{4} \approx 0.7067$