Introduction to Numerical Analysis I
Handout 8

1 Interpolation (cont)

1.8.4 Cubic Spline

Lets next consider a cubic spline. Since S is cubic polynomial of degree on $[x_i, x_{i+1}]$ one writes

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & x \in [x_0, x_1] \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & x \in [x_1, x_2] \\ \cdots & \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 & x \in [x_{n-1}, x_n] \end{cases}$$

Thus we got n polynomials with 4 coefficients each, that is we have $4n$ unknowns. The first $2n$ equations comes from interpolation condition, which also give continuouness on knots:

$$\begin{gathered} S_0(x_0) = f(x_0); \\ S_0(x_1) = f(x_1) = S_1(x_1) \\ \cdots \\ S_{n-2}(x_{n-1}) = f(x_{n-1}) = S_{n-1}(x_{n-1}) \\ S_{n-1}(x_n) = f(x_n) \end{gathered}$$

The other $2(n-1)$ comes from matching the derivatives on knots, i.e. from smoothness, that is

$$\begin{gathered} S_0'(x_1) = S_1'(x_1),\ S_1'(x_2) = S_2'(x_1)\ \cdots\ S_{n-2}'(x_{n-1}) = S_{n-1}'(x_{n-1}) \\ S_0''(x_1) = S_1''(x_1),\ S_1''(x_2) = S_2''(x_1)\ \cdots\ S_{n-2}''(x_{n-1}) = S_{n-1}''(x_{n-1}) \end{gathered}$$

1.8.5 Boundary Conditions

We still need 2 equations which will come from boundary conditions. There is several common boundary conditions for cubic splines.

1. The boundary condition we already saw are periodic $S_0'(x_0) = S_n'(x_n) \quad S_0''(x_0) = S_n''(x_n)$, but they are mostly (if not only) appropriate if the original function is periodic.

2. In the **complete/clamped** cubic spline, the slope conditions $S_0'(x_0) = f'(x_0) \quad S_n'(x_n) = f'(x_n)$ are imposed. These first derivative values of the data may not be readily available but they can be replaced by accurate approximations.

3. The **natural** (or free) boundary condition $S_0''(x_0) = 0 \quad S_n''(x_n) = 0$ The natural spline let the slope at the ends to be free to equilibrate to the position that minimzes oscilatory behaviour of the curve. However, the natural cubic spline is seldom used since it does not provide a sufficiently accurate approximation. One of the reasons is because the value of f'' is not necessary zero, for example $f(x) = x^2$. Note that regular cubic interpolation would produce exact solution to this example.

4. Instead of imposing the natural cubic spline conditions, we could use the correct second derivative values:

   $$S_0''(x_0) = f''(x_0) \quad S_n''(x_n) = f''(x_n)$$

   This options adjusts curvature at end points. These second derivative values of the data are not usually available but they can be replaced by accurate approximations.

5. A simpler, sufficiently accurate spline is determined using the **not-a-knot** boundary condition

   $$S_0'''(x_1) = S_1'''(x_1) \quad S_n'''(x_{n-1}) = S_{n-1}'''(x_{n-1}),$$

   This condition forces that $S_0 \equiv S_1$ and $S_n \equiv S_{n-1}$, because they agree on values of $0 - 3$ derivatives. This make x_1 and x_{n-1} no longer knots.

1.8.6 Construction of cubic spline

We consider here slightly different construction of cubic spline then in the recent example. Since $S_j(x)$ is cubic polynomial it's second derivative $S''_j(x)$ is linear.

Let $S''_j(x_{j+1}) = z_{j+1} = S''_{j+1}(x_{j+1}),\ 0 \le j \le n-1$ then $S''_j(x) = z_j \frac{(x-x_{j+1})}{x_j - x_{j+1}} + z_{j+1}\frac{(x_j - x)}{x_j - x_{j+1}},\ 0 \le j \le n-1$ Integrating twice gives

$$S_j(x) = \frac{z_j}{6}\frac{(x-x_{j+1})^3}{x_j - x_{j+1}} + \frac{z_{j+1}}{6}\frac{(x_j - x)^3}{x_j - x_{j+1}} + \tilde{C}_j x + \tilde{D}_j = \frac{z_j}{6}\frac{(x-x_{j+1})^3}{x_j - x_{j+1}} + \frac{z_{j+1}}{6}\frac{(x_j - x)^3}{x_j - x_{j+1}} + C_j(x_j - x) + D_j(x - x_{j+1})$$

From interpolation condition one gets

$$S_j(x_j) = \frac{z_j}{6}\frac{(x_j - x_{j+1})^3}{x_j - x_{j+1}} + D_j(x_j - x_{j+1}) \qquad = f(x_j)$$

$$S_j(x_{j+1}) = \frac{z_{j+1}}{6}\frac{(x_j - x_{j+1})^3}{x_j - x_{j+1}} + C_j(x_j - x_{j+1}) \qquad = f(x_{j+1})$$

denote $h_j = x_j - x_{j+1}$ and solve it for C and D to get $D_j = \frac{f(x_j)}{h_j} - \frac{z_j h_j}{6}, \quad C_j = \frac{f(x_{j+1})}{h_j} - \frac{z_{j+1}h_j}{6}$

To determine z_i one use $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}),\ 0 \le j \le n-2$ as following

$$S'_j(x) = \frac{z_j}{2}\frac{(x-x_{j+1})^2}{h_j} - \frac{z_{j+1}}{2}\frac{(x_j - x)^2}{h_j} + D_j - C_j = \frac{z_j}{2}\frac{(x-x_{j+1})^2}{h_j} - \frac{z_{j+1}}{2}\frac{(x_j - x)^2}{h_j} + \frac{f(x_j) - f(x_{j+1})}{h_j} - \frac{z_j h_j}{6} + \frac{z_{j+1}h_j}{6}$$

similarly

$$S'_{j+1}(x) = \frac{z_{j+1}}{2}\frac{(x-x_{j+2})^2}{h_{j+1}} - \frac{z_{j+2}}{2}\frac{(x_{j+1} - x)^2}{h_{j+1}} + \frac{f(x_{j+1}) - f(x_{j+2})}{h_{j+1}} - \frac{z_{j+1}h_{j+1}}{6} + \frac{z_{j+2}h_{j+1}}{6}$$

thus

$$S'_j(x_{j+1}) = -\frac{z_{j+1}h_j}{3} + \frac{f(x_j) - f(x_{j+1})}{h_j} - \frac{z_j h_j}{6} \quad \text{and} \quad S'_{j+1}(x_{j+1}) = \frac{z_{j+1}h_{j+1}}{3} + \frac{f(x_{j+1}) - f(x_{j+2})}{h_{j+1}} + \frac{z_{j+2}h_{j+1}}{6}$$

Therefore, for $1 \le j \le n-2$ we have

$$\frac{z_{j+1}h_{j+1}}{3} + \frac{z_{j+1}h_j}{3} + \frac{z_{j+2}h_{j+1}}{6} + \frac{z_j h_j}{6} = \frac{f(x_j) - f(x_{j+1})}{h_j} - \frac{f(x_{j+1}) - f(x_{j+2})}{h_{j+1}}$$

$$z_j h_j + 2z_{j+1}(h_{j+1} + h_j) + z_{j+2}h_{j+1} = 6(f[x_j, x_{j+1}] - f[x_{j+1}, x_{j+2}])$$

The equations for z_0 and z_n we obtain from boundary conditions, for example for complete cubic spline $S'_0(x_0) = f'(x_0)$ and $S'_{n-1}(x_n) = f'(x_n)$ gives

$$S'_0(x_0) = \frac{z_0 h_0}{3} + \frac{f(x_0) - f(x_1)}{h_0} + \frac{z_{j+1}h_0}{6} = f'(x_0) \qquad \text{and} \qquad S'_{n-1}(x_n) = -\frac{z_n h_{n-1}}{3} + \frac{f(x_{n-1}) - f(x_n)}{h_{n-1}} - \frac{z_n h_n}{6} = f'(x_n)$$

$$2z_0 h_0 + z_{j+1}h_0 = 6(f'(x_0) - f[x_0, x_1]) \qquad 2z_n h_{n-1} + z_j h_j = -6(f'(x_n) - f[x_{n-1}, x_n])$$

For an uniform grid $h_j = h$ the resulting linear system reads for

$$A = h\begin{bmatrix} 1 & 2 & & & & \\ 1 & 4 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & 4 & 1 \\ & & & & 2 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 6(f'(x_0) - f[x_0, x_1]) \\ \vdots \\ 6(f[x_j, x_{j+1}] - f[x_{j+1}, x_{j+2}]) \\ \vdots \\ -6(f'(x_n) - f[x_{n-1}, x_n]) \end{bmatrix}$$

1.8.7 Error of cubic spline

The error of (cubic) spline depends on the boundary condition. Given $h = \max_j h_j$ For the natural/free boundary conditions the error comes from the terminal points and is bounded by $O(h^2)$ unless the third derivative is really equals zero at the terminal points. As being said natural spline doesn't reseemble lower order polynomials, which is the case of even a regular interpolation. The other boundary conditions are boundeded by $O(h^4)$. Particularly, it is provable that for clamped/complete cubic spline

$$||f^{(r)}(x) - S^{(r)}(x)|| \le \frac{C_r h^{4-r}}{4!} \max_{c \in [a,b]} f^{(4)}(c), \quad r = 0, 1, 2, 3,$$

where $C_0 = \frac{5}{2^4}$, $C_1 = 1$, $C_2 = 9$ and $C_3 = 12\left(\frac{h}{\min_j h_j} + \frac{\min_j h_j}{h}\right)$.