Introduction to Numerical Analysis I
Handout 7

1 Interpolation (cont)

1.7 Chebyshev Interpolation Points

We would like to minimize the error of interpolation

$$|e(x)| = |f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} p(x) \right|$$

by choosing the interpolation points $x_0, ..., x_n \subset [a,b]$ so that the polynomial $p(x) = \prod_{k=0}^{n} (x - x_k)$ is minimal in the following sense

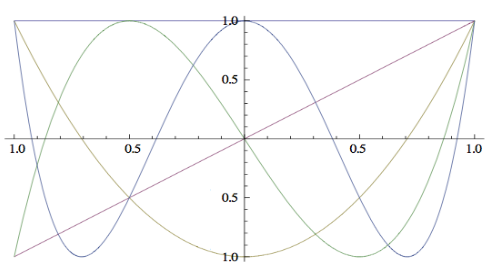
$$\min_{\{x_k\}_{k=0}^n \subset [a,b]} \max_{x \in [a,b]} |p(x)|$$

This is sort of so called MINMAX problem.

Theorem 1.1. The solutions to MINMAX problem at $[a,b] = [-1,1]$, that is

$$\min_{\{x_k\}_{k=0}^n \subset [-1,1]} \max_{x \in [-1,1]} \left| \prod_{k=0}^{n} (x - x_k) \right|$$

are roots of Chebyshev Polynomial $T_{n+1}(x)$.



Definition 1.2 (Chebyshev Polynomials)**.** The Chebyshev Polynomials are given by

$$T_n(x) = \cos(n \arccos x) \text{ or}$$

$$T_n(\cos x) = \cos nx \text{ or}$$

$$T_0 = 1, \ T_1 = x, \ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ n > 1$$

Properties

1. T_{2n} are even functions and T_{2n+1} are odd.
2. Roots of Chebyshev polynomial $T_{n+1}(x)$ are
$$x_k = \cos \frac{\pi}{2} \frac{2k+1}{n+1}, \qquad k = 0, 1, 2, ..., n$$
3. The leading coefficient of $T_{n+1}(x)$ is 2^n whereas $p(x)$ is monic polynomial, therefore
$$p(x) = \frac{1}{2^n} T_n(x).$$
4. Extrema of $T_n(x)$ are at $y_k = \cos \frac{\pi k}{n}$ and $T_n(y_k) = (-1)^k$
5. $|T_n(x)| \leq 1$, therefore $|p(x)| \leq \frac{1}{2^n}$.

1.7.1 Chebyshev points on general interval

Consider $z_k \in [a,b]$. One expresses it using Chebyshev interpolation points $x_k \in [-1,1]$ as

$$z_k = \frac{a+b}{2} + \frac{b-a}{2} x_k$$

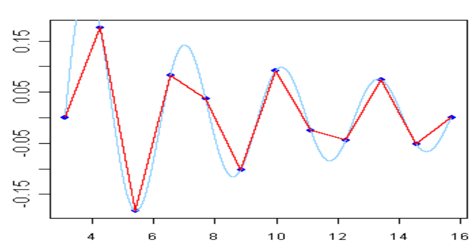
$\rho(z) = \prod_{k=0}^{n} (z - z_k) = \left(\frac{b-a}{2}\right)^{n+1} \frac{1}{2^n} T_{n+1}(x).$

Therefore $|p(z)| \leqslant \left| \left(\frac{b-a}{2}\right)^{n+1} \frac{1}{2^n} \right| = 2\left(\frac{b-a}{4}\right)^{n+1}$

1.8 Spline Interpolation

We would like to interpolate using as many as possible information about function, i.e. values of the function at many points without the risk to increase the error due to possibly unbounded derivatives.

1.8.1 Piecewise Interpolation



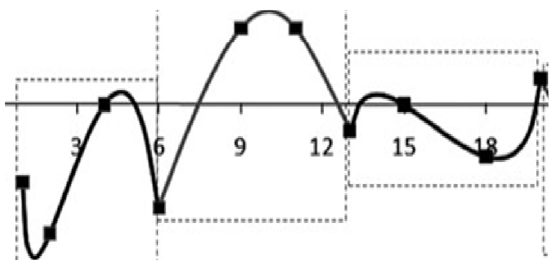
The obvious solution to the problem of error of interpolation is to use only the points nearby to the point of interest. For example

Definition 1.3 (Piecewise Linear Interpolation)**.** Let $f(x)$ be known at points $\{x_k\}_{k=0}^n$, for each $x \in [x_j, x_{j+1}]$ we approximate $f(x_j)$ by a piece wise linear function $g_1(x) =$

$$\begin{cases} P_1^0(x) = f(x_0) + f[x_0,x_1](x - x_0) & x \in [x_0, x_1] \\ P_1^1(x) = f(x_1) + f[x_1,x_2](x - x_1) & x \in [x_1, x_2] \\ \vdots & \\ P_1^{n-1}(x) = f(x_{n-1}) + f[x_{n-1},x_n](x - x_{n-1}) & x \in [x_{n-1}, x_n] \end{cases}$$

The error for $x \in [a,b]$ is given by

$$\begin{aligned}|e(x)| = |f - g_1| &\le \max_j \left\{ \frac{|x_j - x_{j+1}|^2}{2^2} \frac{\max_{x_j \leqslant c \leqslant x_{j+1}} \left|f^{(2)}(c)\right|}{2} \right\} \\ &\le \frac{1}{8} \max_{a \le c \le b} \left|f^{(2)}(c)\right| \max_j |x_j - x_{j+1}|^2\end{aligned}$$



Definition 1.4 (Piecewise Polynomial Interpolation)**.** Let $f(x)$ be known at points $\{x_k\}_{k=0}^n$, for each $x \in [x_{jm}, x_{(j+1)m}]$, where $m \ge 1$, $0 \le j \le \frac{n-m}{m}$ we approximate $f(x_j)$ by

$$g_m(x) = \begin{cases} P_m^0(x) = f[x_0] + \cdots f[x_0, ..., x_m] \prod_{k=0}^{m} (x - x_k) & x \in [x_0, x_m] \\ P_m^1(x) = f[x_m] + \cdots f[x_m, ..., x_{2m}] \prod_{k=m}^{2m} (x - x_k) & x \in [x_m, x_{2m}] \\ \vdots & \\ P_m^{\frac{n-m}{m}}(x) = f[x_{n-m}] + \cdots & x \in [x_{n-m}, x_n] \\ \qquad + f[x_{n-m}, ..., x_n] \prod_{k=n-m}^{n} (x - x_k) & \end{cases}$$

The error is then given by

$$\begin{aligned}&|e\,(x)| = |f(x) - g_m(x)| \le \\ &\le \frac{\max_{a \le c \le b} \left|f^{(m+1)}(c)\right|}{2^{m+1}(m+1)!} \max_j \left|x_{jm} - x_{(j+1)m}\right|^{m+1}\end{aligned}$$

If $m << n$ the error is much better then in equation (1.8.1).

1.8.2 Splines

We would like the interpolation to smooth.

Definition 1.5 (The Spline Interpolation). Let

$$x_0 \le x_1 \le ... \le x_n$$

be interpolation points, also called the **knots** of the spline. A spline function S of degree $k \ge 0$ is a function that satisfies the following

- On each interval x_j, x_{j+1}, S is a polynomial of degree $\le k$.
- $S \in \mathcal{C}^{k-1}$ on $[x_0, x_n]$ (the level of smoothness).

A piecewise linear interpolation is a spline with 0-smoothness.

In order to define a smooth spline one requires that derivatives are continuous at each inner knot, i.e. $S^{(j)}(x_i^+) = S^{(j)}(x_i^-)$. A very special case of spline is when the derivatives are known at knots, in such case the spline is a special case of Hermit interpolation.

1.8.3 Hermit interpolation

Hermit interpolation can be considered a generalization of Newton interpolation. Unlike Newton interpolation, Hermite interpolation matches an unknown function both in observed value, and the observed value of its derivatives.

We proved earlier that $f\,[x_0, \ldots, x_n] = \frac{f^{(n)}(c)}{n!}$ when $x_j \ne x_k$ for all j, k. In case of $x_j = x_k$ we have a removable discontinuity, therefore we can define it as following

Definition 1.6 (Continuous Divided Differences)**.**

$$f[\underbrace{x, ..., x}_{n+1}] = \lim_{\forall j, h_j \to 0} f[x, x+h_1, ..., x+h_n] = \frac{f^{(n)}(x)}{n!}$$

Now Newton method is not limited to the different points, but we need to know derivatives. Given f(x) and it's m_k derivatives at x_k, that is $f\,(x_k)$, $f'\,(x_k)$, ..., $f^{(m_k)}\,(x_k)$. We build the triangular table of divided differences as following: we repeat the x_k and $f(x_k)$ for $m_k + 1$ times. For the repeated points we use known derivatives, otherwise the regular divided differences, for example:

$$\begin{array}{llllll}
x_0 & \underline{f\,[x_0]} & & & & \\
 & & \underline{f'\,(x_0)} & & & \\
x_0 & f\,[x_0] & & \underline{f[x_0, x_0, x_1]} & & \\
 & & \underline{\frac{f[x_1] - f[x_0]}{x_1 - x_0}} & & \underline{f[x_0, x_0, x_1, x_1]} & \\
x_1 & f\,[x_1] & & f[x_0, x_1, x_1] & & \underline{f[x_0, x_0, x_1, x_1, x_1]} \\
 & & f'\,(x_1) & & f[x_0, x_1, x_1, x_1] & \\
x_1 & f\,[x_1] & & f''(x)/2 & & \\
 & & f'\,(x_1) & & & \\
x_1 & f\,[x_1] & & & &
\end{array}$$

Error of Hermit Interpolation is given by

$$e\,(x) = \frac{f^{(2(n+1))}\,(c)}{(2\,(n+1))!} \prod_{k=0}^{n} (x - x_k)^2$$

For a general case, when m_k defivatives of the functions known at point x_k one uses the original Newton's interpolation formula of the error with

$$n + 1 = \sum_k (m_k + 1)$$