Introduction to Numerical Analysis I
Handout 1

1 Calculus Review

Definition 1.1. A function f is continuous at the point x=a if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x in the domain of f with $c - \delta < x < c + \delta$, the value of $f(x)$ satisfies

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon.$$

More simple version: A function f is continuous at the point x=a if $\lim_{x \to a} f(x) = f(a)$.

Definition 1.2. A function f is continuous on the interval I if $\lim_{x \to a} f(x) = f(a)$ for every $x \in I$.

Definition 1.3. I function f is continuous on the interval I then f attain maximum and minimum on I.

Theorem 1.4 (Intermediate Value Theorem). Let function f be continuous on the closed interval I, and let

$$N \in [\min_{x \in I} f(x), \max_{x \in I} f(x)],$$

i.e. N is in the range of f, then there is exists $c \in I$, such that $f(c) = N$.

Theorem 1.5 (Mean Value Theorem). Let function f be continuous on the closed interval $[a, b]$ and differentiable on an open integral (a, b), then there is exists $c \in (a, b)$, such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

Theorem 1.6 (Mean Value Theorem for Integrals). Let function f be continuous on the closed interval $[a, b]$ and let $w(x)$ be non negative and integrable on $[a, b]$, then there is exists $c \in (a, b)$, such that

$$\int_a^b f(x)w(x)dx = f(c) \int_a^b w(x)dx.$$

Note:A more common version of this theorem is given by the particular case of $w(x) = 1$, so that $\int_a^b w(x)dx = \int_a^b dx = b - a$, and so $f(c)$ is an average value of f:

$$f(c) = \frac{1}{b - a} \int_a^b f(x)dx.$$

Definition 1.7 (Taylor's Polynomial & Series/Expansion). Let f have $n + 1$ $(n \geq 0)$ continuous derivatives on $[a, b]$ and let $x, x_0 \in [a, b]$, then Taylor Series are given by

$$f(x) = T_n(x) + R_{n+1}(x),$$

where T_n is the Taylor polynomial of nth order given by

$$T_n(x) = \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$$

and $R_{n+1}(x) = \sum\limits_{j=n+1}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$ is the remainder. It can be proven that there is exists $c \in [a, b]$, such that

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

1.1 Asymptotic Order Notations

Definition 1.8. We denote $f(x) = O(g(x))$ (, i.e. "f is O g") as $x \to a$ and say that $f(x)$ is bounded above by $g(x)$ in the vicinity of a if there is exists numbers M and δ, such that $|f(x)| \leq M\,|g(x)|$ for $x \in (a - \delta, a + \delta)$.

Similarly, $f(x)$ is bounded above by $g(x)$ at infinity, i.e. $f(x) = O(g(x))$ as $x \to \infty$ if there exists numbers M and x_0 such that $|f(x)| \leq M\,|g(x)|$ for $x \geq x_0$.

Example 1.9. Given a polynomial

$$P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n,$$

It is easy to see that $P_n(x) = O(x^n)$.

Example 1.10. Consider $f(a + h)$ is approximated by Taylor polynomial of second order, i.e. T_2, about a, then

$$f(a + h) \approx f(a) + hf'(a) + \frac{h^2}{2} f''(a),$$

where h considered small number. The error term is given by $f(a+h) - f(a) = R_3(a+h) = \frac{h^3}{6} f'''(c)$ for some $c \in (a, a + h)$. Note that $R_3(a + h) = O(h^3)$ as $h \to 0$ and therefore one writes

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + O(h^3)$$