10.2 Derivatives of Elementary functions (3.1):

Ex 1. $f(x)=c;\ f(x)'=\lim\limits_{h\to 0}\frac{f(h+x)-f(x)}{h}=\lim\limits_{h\to 0}\frac{c-c}{h}=0$

Ex 2. $f(x)=x;\ f(x)'=\lim\limits_{h\to 0}\frac{f(h+x)-f(x)}{h}=\lim\limits_{h\to 0}\frac{(h+x)-x}{h}=1$

Ex 3. $f(x)=x^2;\ f(x)'=\lim\limits_{h\to 0}\frac{f(h+x)-f(x)}{h}=\lim\limits_{h\to 0}\frac{(h+x)^2-x^2}{h}=\lim\limits_{h\to 0}\frac{h^2+2xh+x^2-x^2}{h}=$

$=\lim\limits_{h\to 0}(h+2a)=2x$

Ex 4. $f(x)=x^3;\ f'(x)=\lim\limits_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim\limits_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\lim\limits_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=$

$\lim\limits_{h\to 0}3x^2+3xh+h^2=3x^2$

10.2.1 Power Rule

Binomial Thm (found on ref page 1 of the book):

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2}x^{n-2}y^2+\ldots+\frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}x^{n-k}y^k+\ldots+nxy^{n-1}+y^n$$

Def(The Power Rule):

$$f(x)=x^n,\ f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=$$

$$=\lim_{h\to 0}\frac{x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\ldots+\frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}x^{n-k}h^k+\ldots+nxh^{n-1}+h^n-x^n}{h}=$$

$$=\lim_{h\to 0}\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\ldots+\frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}x^{n-k}h^{k-1}+\ldots+nxh^{n-2}+h^{n-1}\right)=nx^{n-1}$$

Ex 5. $\frac{d}{dx}x^{1234567}=1234567x^{1234566}$

Ex 6. $(x^n)^{(n)}=n(x^{n-1})^{(n-1)}\ldots=n\cdots(n-k+1)(x^{n-k})^{(n-k)}=\frac{n!}{(n-k)!}(x^{n-k})^{(n-k)}\ldots=\frac{n!}{\cancel{0!}}\cancel{x^0}$

Ex 7. $f(x)=\frac{1}{x};\ f'(x)=\lim\limits_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right)=\lim\limits_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{(x+h)x}\right)=\lim\limits_{h\to 0}\frac{-1}{(x+h)x}=-\frac{1}{x^2}$ from the other side one can do $\frac{d}{dx}\frac{1}{x}=\left(x^{-1}\right)'=-1\cdot x^{-1-1}=-\frac{1}{x^2}$

Ex 8. $\frac{d}{dx}\sqrt{x}=\left(x^{1/2}\right)'=-\frac{1}{2}x^{1/2-1}=-\frac{1}{2}x^{-1/2}=-\frac{1}{2\sqrt{x}}$

Ex 9. $\frac{d}{dx}\sqrt{\frac{1}{x}}=\lim\limits_{h\to 0}\frac{\sqrt{\frac{1}{x+h}}-\sqrt{\frac{1}{x}}}{h}=\lim\limits_{h\to 0}\frac{\sqrt{\frac{1}{x+h}}^2-\sqrt{\frac{1}{x}}^2}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\lim\limits_{h\to 0}\frac{\frac{x}{(x+h)x}-\frac{x+h}{(x+h)x}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\lim\limits_{h\to 0}\frac{-\frac{h}{(x+h)x}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=-\frac{1}{2x^{3/2}}$

Def: For any real number α, we define $\frac{d}{dx}x^{\alpha}=\alpha x^{\alpha-1}$

10.3 Derivative of $cf(x)$ and $f(x)\pm g(x)$

Consider f(x) is differentiable, then for any constant c, the function g(x)=cf(x) have the following derivative:

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{cf(x+h)-c(x)}{h}=c\lim_{h\to 0}\frac{f(x+h)-(x)}{h}=cf'(x)$$

For any two differentiable functions f(x) and g(x) the following is true:

$$\frac{d}{dx}\{f(x)\pm g(x)\}=\lim_{h\to 0}\frac{(f(x+h)\pm g(x+h))-f(x)\pm g(x)}{h}=$$
$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\pm\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$$

10.4 Derivative of Exponential function a^x (3.1)

Consider $f(x)=a^x$, then

$$f'(x)=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}=a^x\lim_{h\to 0}\frac{a^h-1}{h}=a^x\cdot f'(0)$$

which means a^x is differentiable if and only if it differentiable at x=0. We can't prove the existence of this limit by the tools we have learned by now. The curious can use the table method we learned at the beginning and find out that for several values of a it is certainly exists. The other one can be convinced by observing the graph of the function, there is no corners, kinks, jumps, or sharp change, i.e. the tangent line isn't a vertical line.

The more convenient base *a* will obviously be the one that satisfy $f'(0)=1$. We already have learned such number; this is *e*, which we now can redefine as a number that satisfy $\lim_{h\to 0}\frac{e^h-1}{h}=1$. We actually defined *e* very similar before, since we required the slope of the tangent line at x=0 to be equal 1, but this exactly what we ask here.

We just showed that $\frac{d}{dx}e^x=e^x$.

10.5 The Product Rule (3.2)

Unlike the Sum and Difference Rule $\frac{d}{dx}\{f(x)g(x)\}\neq f'(x)g'(x)$, for example $(x^2)'=(x\cdot x)'\neq x'\cdot x'=1\cdot 1$(but we know that $(x^2)'=2x$). Let us develop the right formula.

Let's f(x) and g(x) be differentiable. Define $\Delta^h f = f(x+h) - f(x)$ and solve for $f(x+h)$ to get $f(x+h) = f(x) + \Delta f$ (from now we'll omit the h in Δ^h).

$$\frac{d}{dx} f(x)g(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\frac{(f(x)+\Delta f)(g(x)+\Delta g) - f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\frac{f(x)g(x) + \Delta g f(x) + \Delta f g(x) + \Delta f\,\Delta g - f(x)g(x)}{h} = \lim_{h\to 0}\frac{\Delta g f(x) + \Delta f g(x) + \Delta f\,\Delta g}{h} =$$

$$f(x)\lim_{h\to 0}\frac{\Delta g}{h} + g(x)\lim_{h\to 0}\frac{\Delta f}{h} + \lim_{h\to 0}\frac{\Delta f\,\Delta g}{h}\cdot\frac{h}{h} = f(x)g'(x) + f'(x)g(x) + \lim_{h\to 0}\frac{\Delta f}{h}\lim_{h\to 0}\frac{\Delta g}{h}\lim_{h\to 0} h =$$

$$= f(x)g'(x) + f'(x)g(x)$$

Ex 1. $(x^2)' = (x\cdot x)' \neq x\cdot 1 + 1\cdot x = 2x$

Ex 2. $\frac{d}{dx}(e^x x^3) = x^3\frac{d}{dx}e^x + e^x\frac{d}{dx}x^3 = x^3 e^x + e^x\cdot 3x^2$

Ex 3. $\frac{d^2}{dx^2}(e^x x^3) = \frac{d}{dx}(x^3e^x + 3x^2e^x) = \frac{d}{dx}(x^3e^x) + \frac{d}{dx}(e^x\cdot 3x^2) = x^3e^x + 3x^2e^x +$
$+3x^2\frac{d}{dx}e^x + e^x\left(\frac{d}{dx}3x^2\right) = x^3e^x + 3x^2e^x + 3x^2e^x + e^x\cdot 6x = x^3e^x + 6x^2e^x + 6xe^x$

Ex 4. $\left[(5+x^3)(2+x)\right]' = (5+x^3)'(2+x) + (5+x^3)(2+x)' = 3x^2(2+x) + (5+x^3) = 4x^3 + 6x^2 + 5$

Ex 5. $\frac{d}{dx}\left[\left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x\right)\right] = (\ )'(\ ) + (\ )(\ )' = \left(-\frac{1}{x^2}+1\right)\left(\frac{1}{x}-x\right) + \left(\frac{1}{x}+x\right)\left(-\frac{1}{x^2}-1\right) =$
$= \frac{-\left(\frac{1}{x}-x\right)^2 - \left(\frac{1}{x}+x\right)^2}{x} = -2\frac{\frac{1}{x^2}+x^2}{x} = \frac{-2}{x^3} - 2x$

Ex 6. $\left(x^4 f(x)\right)'' = \left[4x^3 f + x^4 f'\right]' = \left[12x^2 f + 4x^3 f'\right] + \left[4x^3 f' + x^4 f''\right] = 12x^2 f + 8x^3 f' + x^4 f''$

Ex 7. $\left(x^4 f(x)\right)^{(3)} = \left[12x^2 f + 8x^3 f' + x^4 f''\right]' = 24xf + (12+24)x^2 f' + (8+4)x^3 f'' + x^4 f^{(3)}$

10.6 The Quotient Rule (3.2)

Let's f(x) and g(x) be differentiable.

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} = \frac{1}{g(x)}\left(\lim_{h\to 0}\frac{(f(x)+\Delta f)g(x) - f(x)(g(x)+\Delta g)}{hg(x+h)}\right) =$$

$$= \frac{1}{g(x)}\lim_{h\to 0}\frac{(f(x)g(x) + \Delta f g(x)) - (f(x)g(x) + \Delta g f(x))}{hg(x+h)} = \frac{1}{g(x)}\lim_{h\to 0}\frac{\Delta f g(x) - \Delta g f(x)}{hg(x+h)} =$$

$$=\frac{g(x)}{g(x)}\lim_{h\to0}\frac{\Delta f}{h}\lim_{h\to0}\frac{1}{g(x+h)}-\frac{f(x)}{g(x)}\lim_{h\to0}\frac{\Delta g}{h}\lim_{h\to0}\frac{1}{g(x+h)}=$$

$$=\frac{g(x)}{g(x)}f'(x)\frac{1}{g(x)}-g'(x)\frac{f(x)}{g(x)}\frac{1}{g(x)}=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$$

Ex 1. $\frac{d}{dx}\frac{1}{x}=\frac{x\cdot0-1\cdot1}{x^2}=-\frac{1}{x^2}$

Ex 2. $\frac{d}{dx}\frac{3x^3-2x^2+4x}{x^2-3}=\frac{(x^2-3)(3x^3-2x^2+4x)'-(x^2-3)'(3x^3-2x^2+4x)}{(x^2-3)^2}=$

$$\frac{(x^2-3)(9x^2-4x+4)-2x(3x^3-2x^2+4x)}{(x^2-3)^2}=\frac{3x^4-31x^2+12x-12}{(x^2-3)^2}$$

Ex 3. $\frac{d}{dx}\left[\frac{x^5+x^2}{x^3+3}\right]=\frac{(x^5+x^2)'(x^3+3)-(x^5+x^2)(x^3+3)'}{(x^3+3)^2}=\frac{(5x^4+2x)(x^3+3)-(x^5+x^2)3x^2}{(x^3+3)^2}=$

$$=\frac{5x^7+2x^4+15x^4+6x-(3x^7+3x^4)}{(x^3+3)^2}=\frac{2x^7+14x^4+6x}{x^6+6x^3+9}$$

10.7 Derivatives of Trigonometric functions (3.3)

Before we start with $\frac{d}{dx}\sin x$, let remember that $\lim_{h\to0}\frac{\sin h}{h}=1$ and do

$$\lim_{h\to0}\frac{\cos h-1}{h}\frac{\cos h+1}{\cos h+1}=\lim_{h\to0}\frac{\cos^2 h-1}{h(\cos h+1)}=\lim_{h\to0}\frac{-\sin^2 h}{h(\cos h+1)}=-\lim_{h\to0}\sin h\lim_{h\to0}\frac{\sin h}{h}\lim_{h\to0}\frac{1}{(\cos h+1)}=$$

$$=0\cdot(-1)\cdot\frac{1}{1+1}=0$$

Now we ready:

$$\frac{d}{dx}\sin x=\lim_{h\to0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}=$$

$$=\lim_{h\to0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to0}\frac{\cos x\sin h}{h}=\sin x\lim_{h\to0}\frac{\cos h-1}{h}+\cos x\lim_{h\to0}\frac{\sin h}{h}=$$

$$=\sin x\cdot0+\cos x\cdot1=\cos x$$

Similarly

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h} =$$
$$= \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} = \cos x\cdot 0 - \sin x \cdot 1 = -\sin x$$

Finally, more interesting:

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + tan^2 x = \frac{1}{\cos^2 x} = \sec^2 x$$

10.8 The Chain Rule (3.4)

$$\frac{d}{dx}f\left(g(x)\right) = \lim_{h\to 0}\frac{f\left(g(x+h)\right) - f\left(g(x)\right)}{h} = \lim_{h\to 0}\frac{f\left(g(x)+\Delta^h g\right) - f\left(g(x)\right)}{h}\cdot\frac{\Delta^h g}{\Delta^h g} =$$
$$= \lim_{h\to 0}\frac{f\left(g(x)+\Delta^h g\right) - f\left(g(x)\right)}{\Delta^h g}\lim_{h\to 0}\frac{\Delta^h g}{h} = f'\left(g(x)\right)g'(x)$$

Thus we got: $\frac{d}{dx}\underbrace{f}_{\substack{outer\\ function}}\left(\underbrace{g(x)}_{\substack{inner\\ function}}\right) = \underbrace{f'\left(g(x)\right)}_{\substack{derivative\,of\\ outer\ function\\ evaluated\ at\,inner\\ function}}\underbrace{g'(x)}_{\substack{derivative\,of\\ inner\\ function}}$

10.8.1 Combination of Power Rule with the chain rule

$$\frac{d}{dx}\left\{g(x)^n\right\} = ng(x)^{n-1}g'(x)$$

Quotient Rule using Product and Chain Rules:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx}f(x)g(x)^{-1} = g(x)^{-1}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)^{-1} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Derivative of exponential function a^x

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\frac{d}{dx}\left\{x\ln a\right\} = e^{\ln a^x}\ln a\frac{d}{dx}x = a^x\ln a$$

Ex 1.

$$\frac{d}{dx}\left[\left(\frac{s}{1+s^2}\right)^{12}\right] = \left[\left(x^{12}\right)\circ\left(\frac{s}{1+s^2}\right)\right]' = \left(12x^{11}\right)\circ\left(\frac{s}{1+s^2}\right)*\left(\frac{s}{1+s^2}\right)' =$$
$$= 12\left(\frac{s}{1+s^2}\right)^{11}\frac{1+s^2-2s^2}{1+s^2} = 12\frac{1-s^2}{1+s^2}\left(\frac{s}{1+s^2}\right)^{11}$$

Ex 2.

$$\frac{d}{dx}\sqrt{\frac{x^3-2}{x+1}}=\frac{d}{dx}\left(\left(\sqrt{x}\right)\circ\frac{x^3-2}{x+1}\right)=\frac{1}{2\sqrt{\frac{x^3-2}{x+1}}}\cdot\frac{3x^2\left(x+1\right)-\left(x^3-2\right)}{\left(x+1\right)^2}=$$

$$=\frac{\sqrt{x+1}}{2\sqrt{x^3-2}}\cdot\frac{2x^3+3x^2+2}{\left(x+1\right)^2}=\frac{2x^3+3x^2+2}{\sqrt{x^3-2}\left(x+1\right)^{3/2}}$$

Ex 3.

$$\frac{d}{dx}\sqrt{\frac{\cos\nu}{\tan\nu+1}}=\frac{d}{dx}\sqrt{\frac{\cos\nu}{\frac{\sin\nu}{\cos\nu}+1}}=\frac{d}{dx}\sqrt{\frac{\cos^2\nu}{\sin\nu+\cos\nu}}=$$

$$\frac{1}{2\sqrt{\frac{\cos^2\nu}{\sin\nu+\cos\nu}}}\cdot\frac{-2\sin\nu\cos\nu\left(\sin\nu+\cos\nu\right)-\cos^2\nu\left(\cos\nu-\sin\nu\right)}{\left(\sin\nu+\cos\nu\right)^2}=$$

$$=\frac{\sqrt{\sin\nu+\cos\nu}}{2\left|\cos\nu\right|}\cdot\frac{-2\left(\sin^2\nu\cos\nu+\cos^2\nu\sin\nu\right)+\cos^3\nu-\cos^2\nu\sin\nu}{\left(\sin\nu+\cos\nu\right)^2}=$$

$$=\text{sign}\left(\cos\nu\right)\frac{\cos^2\nu-2\sin^2\nu-3\cos\nu\sin\nu}{2\left(\sin\nu+\cos\nu\right)^{3/2}}$$

10.8.2 Tangent lines to Parametric Curves

Recall: parametric curves are given by a pair $\left(x,y\right)=\left(f\left(t\right),g\left(t\right)\right)$**.**

Suppose f(t), g(t) are differentiable. Would we able to write the curve in a form y=F(x) we could simply find the slope of the tangent line by using $m=\frac{dy}{dx}=F'\left(x\right)$. Let us assume for a moment that such F exists and use the parameters. We get

$$g\left(t\right)=y=F\left(x\right)=F\left(f\left(t\right)\right)$$

If we derive the outer part of this equation ($g\left(t\right)=F\left(f\left(t\right)\right)$) to get

$$g'\left(t\right)=F'\left(f\left(t\right)\right)f'\left(t\right)$$

we actually get

$$\frac{dy}{dt}=g'\left(t\right)=F'\left(f\left(t\right)\right)f'\left(t\right)=F'\left(z\right)\frac{dx}{dt}=\frac{dy}{dx}\frac{dx}{dt},$$

assuming that $\frac{dx}{dt}\neq 0$,we arrive at $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dt}\cdot\frac{dt}{dx}$.

Def: A line normal to a curve at x=a is a line perpendicular line to the tangent line to the curve at x=a. For example, if y=mx+b is tangent to a curve, then y=-x/m+c is it's normal.

Ex 1. Find a tangent and normal line to $(x,y)=(2\sin t, 3\cos t)$ at $t=\frac{\pi}{4}$:

$m=\frac{dy}{dx}=\frac{-3\sin t}{2\cos t}_{x=\pi/4}=\frac{-3\cdot 1/\sqrt{2}}{2\cdot 1/\sqrt{2}}=-\frac{3}{2}$ and we arrive at tangent line of the form $y=-\frac{3}{2}x+b$, in order to determine b we need to make this line to go through the point $(x,y)=\left(2\sin\frac{\pi}{4}, 3\cos t\frac{\pi}{4}\right)=\left(\frac{2}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$, i.e.

$$\frac{3}{\sqrt{2}}=y=-\frac{3}{2}\frac{2}{\sqrt{2}}+b\Rightarrow b=\frac{3}{\sqrt{2}}+\frac{3}{2}\frac{2}{\sqrt{2}}=\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{2}}=\frac{6}{\sqrt{2}}$$

Similarly one find a normal line by using the form $y=\frac{2}{3}x+c$ and substituting

$$\frac{3}{\sqrt{2}}=y=\frac{2}{3}\frac{2}{\sqrt{2}}+c\Rightarrow c=\frac{3}{\sqrt{2}}-\frac{2}{3}\frac{2}{\sqrt{2}}=\frac{9-4}{3\sqrt{2}}=\frac{5}{3\sqrt{2}}$$