10 Derivatives (2.6-2.8)

The tangent line to the curve y=f(x) at the point (a,f(a)) is the line $l(x) = mx + b$ through this point with slope $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided that this limit exists. Alternatively one can express the slope as $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Instantaneous vs. Average Velocity: If you drive a car from SLC to Las Vegas, your average velocity is given by $\frac{\text{displacement}}{\text{time}} = \frac{f(t+h) - f(t)}{h}$ (note that velocity mean speed and direction, i.e. it can be either negative or positive). However when you ride the car your speedometer shows you different values of the speed\velocity during the trip, because it represents an instantaneous velocity, which can be expressed by $v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

Rate of change: denote change in x as $\Delta x = x_1 - x_2$ and change in y as $\Delta y = y_1 - y_2 = f(x_1) - f(x_2)$, then the rate of change is given $\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$ and instantaneous rate of change is $\lim_{x_1 \to x_2} \frac{f(x_1) - f(x_2)}{x_1 - x_2}$

10.1 The derivative (2.7)

Def: The Derivative of f(x) at x=a is denoted as $f'(a) = \left. \frac{df(x)}{dx} \right|_{x=a}$. $f'(a)$ is an instantaneous rate of change if y=f(x) with respect to x when x=a, i.e. $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided the limit exists. It can be of course alternatively defined as $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h}$ and even $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}$.

Ex 1. Find an angle between $f(x) = 8 - x,\ g(x) = 4\sqrt{x+4}$ at intersection point.

We first find the intersection $4\sqrt{x+4} = 8 - x \Rightarrow 16(x+4) = 64 - 16x + x^2 \Rightarrow 32x = x^2$, therefore $x = 0, 32$, but since $8 - 32 < 0$ the only intersection is at x=0. The angle between a line $y = mx + b$ and X-axis is given by

$$\arctan m = \arctan y',$$

i.e. the angle doesn't depend on b (why?). The angle between function f(x), that is not necessary a line, at x=a and X-axis is the angle of the tangent line of f(x) at x=a. The formula of the tangent line is given by $y = f'(a)x + b$. The angle between 2 lines is the difference between their angles to X-axis. Therefore we first derive f(x) and g(x):

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{(8-x)-(8-a)}{x-a} = \lim_{x\to a}\frac{a-x}{x-a} = \lim_{x\to a}(-1) = -1$$

$$g'(a) = \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{4\sqrt{x+4}-4\sqrt{a+4}}{x-a} = 4\lim_{x\to a}\frac{\left(\sqrt{x+4}-\sqrt{a+4}\right)\left(\sqrt{x+4}+\sqrt{a+4}\right)}{(x-a)\left(\sqrt{x+4}+\sqrt{a+4}\right)} =$$

$$= 4\lim_{x\to a}\frac{(x+4)-(a+4)}{(x-a)\left(\sqrt{x+4}+\sqrt{a+4}\right)} = 4\lim_{x\to a}\frac{x-a}{(x-a)\left(\sqrt{x+4}+\sqrt{a+4}\right)} = 4\lim_{x\to a}\frac{1}{\left(\sqrt{x+4}+\sqrt{a+4}\right)} =$$

$$= \frac{4}{\left(\sqrt{a+4}+\sqrt{a+4}\right)} = \frac{4}{2\sqrt{a+4}} = \frac{2}{\sqrt{a+4}}$$

We next evaluate it at x=0: $f(x) = -1, g(x) = 1$. And finally

$$\theta = \arctan 1 - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Def.:If we allow the number to vary, we can redefine the derivative as a function

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Def.: A function f is **differentiable at x=a** if f'(a) exists. It is **differentiable on an open interval** (a,b) (can also be half- inf or inf) if it is differentiable at every $x \in (a,b)$.

Thm: If f is differentiable at a then f is continuous at a.

Proof:

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(f(a) + f(x) - f(a)\right) = \lim_{x\to a} f(a) + \lim_{x\to a}\left(f(x)-f(a)\right) = f(a) +$$

$$+\lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a) = f(a) + \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a) = f(a) + f'(a)\cdot 0 = f(a)$$

Ex 2. $f = \begin{cases} x^2 & x > 0 \\ -x^2 & x \le 0 \end{cases}$ is differentiable on R including x=0 since

$$f'(0) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = \lim_{h\to 0}\frac{(0+h)^2-\left(-(0-h)^2\right)}{2h} = \lim_{h\to 0}\frac{2h^2}{2h} = \lim_{h\to 0} h = 0$$

10.1.1 Not differentiable functions:

In general a function fails to be differentiable when it discontinuous or have a "corner" or "kink", or have a vertical tangent line at x=a i.e. $\lim_{x\to a} f'(x)=\infty$ (like in $f(x)=\sqrt{x}, \quad f'(x)=\frac{1}{2\sqrt{x}}$ at x=0) .

Important note, that despite that every differentiable is continuous, not every continuous function is differentiable

Ex 3. $f(x)=|x|$ it is continuous at x=0 since $\lim_{x\to 0}|x|=0=f(0)$, but

$$\lim_{h\to 0}\frac{|x+h|-|x|}{h}\underset{x>0}{=}\lim_{h\to 0}\frac{(x+h)-x}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

$$\lim_{h\to 0^-}\frac{|x+h|-|x|}{h}\underset{x<0}{=}\lim_{h\to 0}\frac{-(x+h)-x}{h}=\lim_{h\to 0}\frac{-h}{h}=\lim_{h\to 0}(-1)=-1$$

therefore the derivative doesn't exists.

Ex 4. $f(x)=|x-3|$ is differentiable at any x but x=3

$$\lim_{h\to 0^+}\frac{|3+h-3|-|3-3|}{h}=\lim_{h\to 0^+}\frac{|h|}{h}=1\neq -1=\lim_{h\to 0^-}\frac{|-h|}{h}=\lim_{h\to 0^-}\frac{|3-(h+3)|-|3-3|}{h}=\lim_{h\to 0^-}$$

Ex 5. Find a necessary and sufficient condition for $g(x)=|x-a|\,f(x)$ to be differentiable at x=a.

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=\lim_{x\to a}\frac{|x-a|\,f(x)-|a-a|\,f(a)}{x-a}\underset{\substack{\text{wrong}\\ \text{only for}\\ \text{explanation}}}{=} f(a)\lim_{x\to a^{\pm}}\frac{|x-a|}{x-a}\underset{\substack{\text{means}\\ \text{one sided}\\ \text{derivative}}}{=}\pm f(a)=L$$

$$\exists L \Leftrightarrow f(a)=-f(a)\Leftrightarrow f(a)=0$$

Def.: A **Necessary Condition** for some statement S is a condition that must be satisfied in order for S to obtain.

Def.: A **Sufficient Condition** for some statement S is a condition that, if satisfied, guarantees that S obtains.

Ex 4. For differentiability of function f(x), continuity is necessary condition; however it is not sufficient condition, since some continuous functions aren't differentiable.

Ex 5. For continuity of function f(x), differentiability is a sufficient condition, however it is not necessary condition, and since there are a non-differentiable continues functions.

10.1.2 Higher Derivatives

Since the derivative of a function f(x) is a function f'(x) we can derive it again (given it is differentiable). We denote following derivation as

$$f'(x)=\frac{\partial}{\partial x}f(x)=\frac{\partial f(x)}{\partial x}$$

$$f''(x)=\left(\frac{\partial}{\partial x}\right)^2 f(x)=\frac{\partial^2}{\partial x^2}f(x)=\frac{\partial^2 f(x)}{\partial x^2}=\frac{\partial}{\partial x}f'(x)$$

$$f^{(n)}(x)=\left(\frac{\partial}{\partial x}\right)^n f(x)=\frac{\partial^n}{\partial x^n}f(x)=\frac{\partial^n}{\partial x^n}f(x)=\frac{\partial}{\partial x}f^{(n-1)}(x)$$

Ex 6.

$$\left(2x^2+x-1\right)'=\lim_{h\to 0}\frac{\left(2(x+h)^2+(x+h)-1\right)-\left(2x^2+x-1\right)}{h}=$$

$$=\lim_{h\to 0}\frac{\left(2x^2+4xh+2h^2+x+h-1\right)-\left(2x^2+x-1\right)}{h}=$$

$$=\lim_{h\to 0}\frac{4xh+2h^2+h}{h}=\lim_{h\to 0}4x+2h+1=4x+1$$

$$\left(2x^2+x-1\right)''=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}=\lim_{h\to 0}\frac{\left(4(x+h)+1\right)-\left(4x+1\right)}{h}=\lim_{h\to 0}\frac{4h}{h}=4$$

10.1.3 What does f', f''say about f? (2.8)

Derivatives are very important thing in calculus. One can learn a lot about function f(x) using information about its derivatives. For example the sign of f' provides us information about direction of the function:

If $f'(x)>0$ **on some interval, then f is increasing there.**

If $f'(x)<0$ **on some interval, then is decreasing there.**

Furthermore, the second derivative can provide even more information:

If $f''(x) > 0$ on some interval, then f is concave upward (happy smile) there.

If $f''(x) < 0$ on some interval, then f is concave downward (convex, sad smile) there.

Def (Even function): f(x)=f(-x), i.e. it is symmetric with respect to the y-axis, thus the graph remains unchanged after reflection about the y-axis.

Ex 7. cos x, |x|, x^2

Def (Odd function): f(x) = -f(-x) i.e. it has rotational symmetry with respect to the origin, thus the graph remains unchanged after rotation of 180 degrees about the origin.

Ex 8. x, x^3, sin x

Ex 9. The derivative of even function is odd function and vice verse

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \frac{f(x-h)-f(x)}{h} \underset{h=-t}{=} \lim_{t\to 0}\frac{f(x+t)-f(x)}{-t} = -f'(x)$$

$$g'(-x) = \lim_{h\to 0}\frac{g(-x+h)-g(-x)}{h} = \frac{-g(x-h)+g(x)}{h} \underset{h=-t}{=} \lim_{t\to 0}\frac{g(x+t)-g(x)}{t} = g'(x)$$

Antiderivatives: Some time we know f(x) and we need to find another function F(x) such that F'(x)=f(x). If such F(x) exists we call it antiderivative. Since f(x) is derivative of F(x) we have lot information about F(x) to work with. We'll learn it in future.