8 Continuity(2.4)

Def: A function f is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$

Note: The definition is implicitly require that x=a is in domain of definition of f(x) and that the limit is defined and equal f(a).

Def: A function f is continuous on interval I if $\lim_{x\to a} f(x) = f(a)$ for every $a \in I$.

Note: We understand the interval I as either

- finite: $I=[b_1,b_2]$, $I=[b_1,b_2)$, $I=(b_1,b_2]$, $I=(b_1,b_2)$,
- half finite: $I=[b_1,\infty)$, $I=(-\infty,b_2]$, $I=(b_1,\infty)$, $I=(-\infty,b_2)$
- infinite interval: $I=(-\infty,\infty)$.

In case of finite\ half finite interval we understand the definition as a one sided limit $\lim_{x\to a^-} f(x) = f(a)$ or $\lim_{x\to a^+} f(x) = f(a)$.

Intuitively we understand continuity as no jump, no hole, no tear or break in the graph of the function.

Theorem: Let f(x), g(x) be continuous at x=a, and let c be a constant. The following functions are continuous: $f(x)\pm g(x)$, $cf(x)$, $f(x)g(x)$ and $\frac{f(x)}{g(x)}$ if $g(a)\neq 0$

Corollary:

- Any polynomial is continuous function on $\mathbb{R}=(-\infty,\infty)$
- Any rational function is continuous on its domain of definition.

Theorem: A following functions are continuous on their domain of definition:

- *root functions* - *trigonometric functions*

- *exponential functions* - *logarithmic functions*

Theorem: If f(x) is contiuous at x=b and $\lim_{x\to a} g(x) = b$ then

$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$.

Corollary: If g(x) is continuous at x=a and f(x) is continuous at x=g(a), the the composition function $(f\circ g)(x) = f(g(x))$ is continues at x=a;

Ex 1. $\lim\limits_{x\to -0.5}\frac{2x+1}{|2x+1|} \underset{\substack{f(x)=2x+1 \\ \lim\limits_{x\to -0.5} f(x)=0}}{=} \lim\limits_{f(x)\to 0}\frac{f(x)}{|f(x)|} \underset{y=f(x)}{=} \lim\limits_{y\to 0}\frac{y}{|y|} \Rightarrow$ not exists

DEF (Classification of discontinuities):

1) **Removable discontinuity**: One sided limits at x=a exists, finite and equal, but not equal to f(a) (this is a function with hole, which can be removed, aka "filled").
2) **Step/Jump discontinuity**: One sided limits at x=a are exists and finite **but not equal**.
3) **Essential discontinuity**: At least one of one sided limits not exists or infinite.

Ex 2. $\frac{1}{x}$ and $\sin\frac{1}{x}$ have essential discontinuities at 0.

Ex 3. But $x\sin\frac{1}{x}$ have removable discontinuity because this is a case of bounded function times zero, i.e. the limit at x=0 exists and equal 0.
Another reasoning is since $-1\le \sin\frac{1}{x}\le 1$ we get $-x\le x\sin\frac{1}{x}\le x$, but $\lim\limits_{x\to 0}\pm x=0$, therefore by squeeze theorem $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$.

Ex 4. $\frac{\sqrt{x}-1}{x-1}$ have removable discontinuity at x=1 since $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\frac{1}{2}$(see above)

Ex 5. $\frac{\sin x}{x}$ have removable discontinuity at x=0 (why?)

Ex 6. Let see that $f=\begin{cases} x^2-3 & x\ge 2 \\ \frac{x}{2} & x<2 \end{cases}$ is continuous at x=2.

$$\left.\begin{array}{l} \lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\left(x^2-3\right)=4-3=1 \\ \lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\frac{x}{2}=1 \end{array}\right\}\Rightarrow \lim\limits_{x\to 2} f(x)=1$$

Ex 7. Let see where $f(x)=\begin{cases} 2 & -\infty<x\le 1 \\ 4-2x & 1<x\le 2.5 \\ 2x-7 & 2.5<x<\infty \end{cases}$ is continuous. The function f(x)

Ex 8. combined from continuous function, therefore only x=1 and x=2.5 need to be tested. Let check the limits at these values.

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(4-2x) = 4-2 = 2 = f(1)$$
$$\lim_{x\to2.5^-} f(x) = \lim_{x\to2.5^-}(4-2x) = 4-5 = -1$$
$$\lim_{x\to2.5^+} f(x) = \lim_{x\to2.5^+}(2x-7) = 5-7 = -2 \neq f(2.5)$$

And we found that f(x) is continuous everywhere but at x=2.5 where it has jump discontinuity.

Ex 9.
$$\lim_{t\to0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to0}\frac{\sin 6t}{6t}\frac{3}{\cos 6t}\frac{2t}{\sin 2t} = \lim_{t\to0}\frac{\sin 6t}{6t}\lim_{t\to0}\frac{3}{\cos 6t}\lim_{t\to0}\frac{2t}{\sin 2t} =$$
$$= \lim_{6t=x\to0}\frac{\sin x}{x}\cdot\frac{3}{\cos 0}\cdot\lim_{2t=y\to0}\frac{y}{\sin y} = 1\cdot3\cdot1 = 3$$

Theorem(The intermediate value theorem): Let f(x) be continuous function on closed interval $I=[a,b]$ and let $N\in[f(a),f(b)]$ (or $N\in[f(b),f(a)]$), there is exists $c\in[a,b]$ such that $f(c)=N$.

Ex 10. Show that there is a value $c\in[-\pi,\pi]$ such that $f(x)=\sin x - x$ gets value 1.1: $\sin\pm\pi\mp\pi=\mp\pi$

Ex 11. Does the equations $x^2-5x+1=0$ has a solution:
It is continues and get either negative ($1^2-5\cdot1+1<0$) and positive ($5^2-5\cdot5+1>0$) values, therefore by the intermediate value theorem it should pass through zero, i.e. the solution exists.

We did the following 2 in the class:

Ex 12. Where there following function is continues?

$$f(x)=\begin{cases} x & x<-1 \\ |x| & -1\le x\le 2 \\ x & 2<x\le3 \\ 2x & 3<x\le4 \\ \dfrac{x^2+2x-8}{|x-2|} & x>4 \end{cases}$$

Brief solution: f is piecewise defined function, it continuous at every piece except may be at the connections between pieces, because every piece is known continues function. Therefore the main work here is to see what happens at x=-1, 2, 3, and 4. Thus, we need to check if the limits at these points equal to the function values at these points (aka if $\lim_{x\to a} f(x) = f(a)$). More precisely we need to see if one sided limits are exists and equal at x=-1, 2, 3, and 4.

I won't write down the full solution here (we did it in the class) , verify that you understand why it is continues at x=2 and x=4 and that you know to show there is a jump at x=-1 and at x=3. Note, also that the absolute value here doesn't require special attention since for x>4 we can simply omit it (i.e. change it with x-2)

Ex 13. Does the function $f(x) = \frac{x^2+2x-8}{|x-2|}$ is continuous at x=2

Solution: it is not since the once sided limits aren't equal

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2+2x-8}{|x-2|} = \lim_{x\to 2}\frac{(x-2)(x+4)}{|x-2|}$$

$$\lim_{x\to 2^+}\frac{(x-2)(x+4)}{|x-2|} = \lim_{x\to 2^+}\frac{(x-2)(x+4)}{x-2} = \lim_{x\to 2^+}(x+4) = 6$$

$$\lim_{x\to 2^-}\frac{(x-2)(x+4)}{|x-2|} = \lim_{x\to 2^-}\frac{(x-2)(x+4)}{-(x-2)} = -\lim_{x\to 2^-}(x+4) = -6$$