6 Tangent and Velocity(2.1)

Self reading of chapter 2.1

7 Limit of Function(2.2)

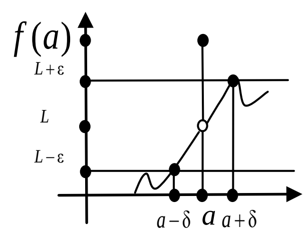
Ex 1. Consider function $f(x)=\frac{x^2-3x+2}{x-1}$. The function f(x) is definitely undefined for x=1. Let's investigate the behavior of f(x) for values near x=1. The following table shows values f(x) for values of x close to but not equal to 1.

x	0.8	0.9	0.99	.999	1.001	1.01	1.1	1.2
f(x)	-1.2	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9	-0.8

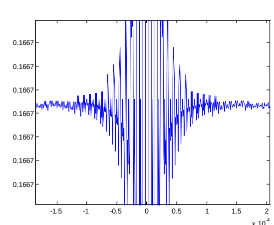
One can see from the table that f(x) getting close to -1 when x approaches 1. For a simple example like this, it is possible to get the same result algebraically since $\frac{x^2-3x+2}{x-1}=\frac{(x-1)(x-2)}{x-1}=x-2$, so this is a line with a hole in x=1.

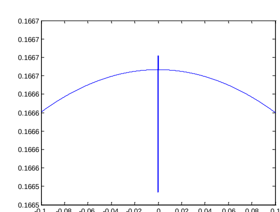
We shall write $\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-3x+2}{x-1}=\lim_{x\to 1}\frac{(x-1)(x-2)}{x-1}=\lim_{x\to 1} x-2=-1$

Def: The equation $\lim_{x\to a} f(x)=L$ means: "the limit of f(x) as x approaches a equals L". Note, the function f(x) isn't required to be defined at x=a. It just has to get closer to L as x gets closer (but not equal) to a.



Note: A more rigorous (ε,δ)-definition of limits talking about existence of an open interval $I_a=(a-\delta,a+\delta)$ for any choice of arbitrary small interval $I_L=(L-\varepsilon,L+\varepsilon)$ (where both $\varepsilon,\delta>0$), such that $\forall x\in I_a$ imply $f(x)\in I_L$.





Ex 2. Let $f(x)=\frac{\sqrt{x^2+9}-3}{x^2}$, find $\lim_{x\to 0} f(x)$

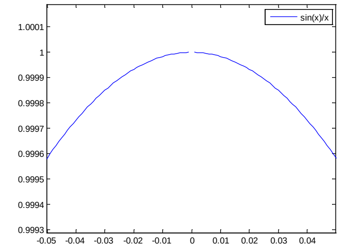
The algebraic approach gives

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}=\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3}=\frac{1}{6}$$

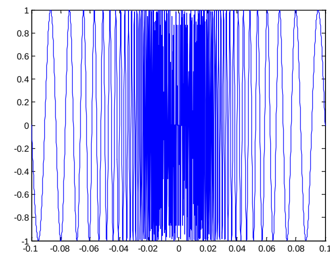
However when one uses a table for really small values of x we may arrive in the wrong answer (this is problem of computer accuracy, in numerator we get 3.0…0 -3, so it considered zero). See also the 2 fine scaled graphs of the function above.

x	±5.E-06	±1.E-06	±5.E-07	±1.E-07	-5.E-08	-1.E-08	1.E-09	5.E-09
f(x)	0.1667	0.1665	0.1670	0.1776	0.1776	0.0000	0.0000	0.0000

Some limits are easy to get from table or sketch like even non polynomial ones like $\lim_{x\to 0}\frac{\sin x}{x}=1$ (see the image), but some may be tricky and some not exists, and some may mislead like in following example.

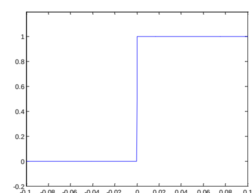


Ex 3. If we create a table to find $\lim_{x\to 0}\sin\frac{\pi}{x}$, one get that for $x=\frac{1}{n}$ for any integer n>0, one gets $\sin n\pi=0$ which can mislead to the **WRONG** guess that $\lim_{x\to 0}\sin\frac{\pi}{x}\underset{wrong}{=}0$. This is wrong because for example for $x=\frac{2}{10^n+1}$ one gets $\sin\left(10^n+1\right)\frac{\pi}{2}=1$ and for $x=\frac{2}{3\left(10^n+1\right)}$ one gets $\sin\left(10^n+1\right)\frac{3\pi}{2}=-1$. Actually the function $\sin\frac{\pi}{x}$ is highly oscillating near x=0 and therefore the limit doesn't exist.



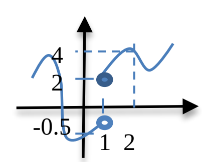
Def: One Sided Limits: The equation $\lim_{x\to a^+}f(x)=L$ means: "the limit of f(x) as x approaches a **from the right** equals L". Similarly $\lim_{x\to a^-}f(x)=L$ means: "the limit of f(x) as x approaches a **from the left** equals L".

Ex 4. For a Heaviside function $f(x)=\begin{cases}0 & x<0\\ 1 & x\ge 0\end{cases}$ the limit $\lim_{x\to 0}f(x)$ isn't exists, because of the jump. However $\lim_{x\to 0^-}f(x)=0$ and $\lim_{x\to 0^+}f(x)=1$. This gives a hint for the following theorem.



Theorem: A limit $\lim_{x\to a}f(x)$ is exists if and only if one sided limits exists and equal, i.e. $\lim_{x\to a^-}f(x)=L=\lim_{x\to a^+}f(x)$.

Ex 5. In the f(x) func on the graph $\lim_{x\to 1^-}f(x)=0.5\neq 2=\lim_{x\to 1^+}f(x)$, therefore $\lim_{x\to 1}f(x)$ not exists (there is a jump), but $\lim_{x\to 2^-}f(x)=4=\lim_{x\to 2^+}f(x)\Rightarrow\lim_{x\to 2}f(x)=4$.



Ex 6. $\lim\limits_{x\to a} c = c;\quad \lim\limits_{x\to a} x = a;\quad \lim\limits_{x\to a} x^2 = a^2;\quad \lim\limits_{x\to a}\frac{1}{x} = \frac{1}{a}$

Ex 7. Let $g(x) = f(x+c)$. Show that if $\lim\limits_{x\to c} f(x) = L$ then $\lim\limits_{x\to 0} g(x) = L$.

Solution: $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} f(x+c) = \lim\limits_{x+c=y\to c} f(y) = L$

7.1 Limit Laws(2.3)

Def: Let c be a constant and $\lim\limits_{x\to a} f(x), \lim\limits_{x\to a} g(x)$ exists and finite. Then

1) $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
2) $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$
3) $\lim\limits_{x\to a} f(x)g(x) = \lim\limits_{x\to a} f(x)\lim\limits_{x\to a} g(x)$
4) $\lim\limits_{x\to a}\frac{f(x)}{g(x)} \underset{\lim_{x\to a} g(x)\neq 0}{=} \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

Ex 8. $n\in\mathbb{Z}^+, \lim\limits_{x\to a} x^n \underset{LL3}{=} \lim\limits_{x\to a} x \lim\limits_{x\to a} x^{n-1} \underset{LL3}{=} \lim\limits_{x\to a} x \lim\limits_{x\to a} x \lim\limits_{x\to a} x^{n-2} = \left(\lim\limits_{x\to a} x\right)^2 \lim\limits_{x\to a} x^{n-2} = \cdots = \left(\lim\limits_{x\to a} x\right)^n = a^n$

Ex 9. Similarly $n\in\mathbb{Z}^+, \lim\limits_{x\to a} f^n(x) \underset{LL3}{=} \left(\lim\limits_{x\to a} f(x)\right)^n$

Ex 10. Evidence $\lim\limits_{x\to 3} x = 3$ and $\lim\limits_{x\to 3} x^3 = 3^3 = 27$, see it on graph of x^3.

Ex 11. Another evidence $\lim\limits_{x\to 3\pi/2}\sin^2 x = \left(\lim\limits_{x\to 3\pi/2}\sin x\right)^2 = (-1)^2$

Ex 12. $n\in\mathbb{Z}^+, \lim\limits_{x\to a}\sqrt[n]{x} = \sqrt[n]{\left(\lim\limits_{x\to a}\sqrt[n]{x}\right)^n} \underset{LL3}{=} \sqrt[n]{\lim\limits_{x\to a}\left(\sqrt[n]{x}\right)^n} = \sqrt[n]{\lim\limits_{x\to a} x} = \sqrt[n]{a}$, for even n assume $a>0$

Ex 13. Similarly $n\in\mathbb{Z}^+, \lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, for even n assume $\lim\limits_{x\to a} f(x) > 0$

Def: Direct Substitution Property: if f(x) "**have no problem at *a***" then $\lim\limits_{x\to a} f(x) = f(a)$. For now we consider that roots, polynomial, rational and trigonometrical functions "**have no problem at *a***" if defined at ***a***.

Ex 14. $\lim\limits_{x\to 5}\frac{(x-2)(x-3)}{x} = \frac{(5-2)(5-3)}{5} = \frac{3\cdot 2}{5} = \frac{6}{5}$

Def: if $f(x) = g(x)$ in an open interval $x\in(a-b,a)\cup(a,a+b)$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$

Ex 15. $\lim\limits_{x\to0}\frac{(x-5)^2-25}{x}=\lim\limits_{x\to0}\frac{x^2-10x+25-25}{x}=\lim\limits_{x\to0}\frac{x^2-10x}{x}=\lim\limits_{x\to0}x-10=-10$

Ex 16. $\lim\limits_{x\to2}\frac{(x-2)(x-3)}{x-2}\neq\frac{(0-2)(0-3)}{0-2}=\frac{"0"}{"0"}$ however $\lim\limits_{x\to2}\frac{(x-2)(x-3)}{x-2}\lim\limits_{x\to2}=(x-3)=0-3=-3$

Ex 17. $\lim\limits_{x\to2}\frac{x^2-3x+2}{x^2-4}=\lim\limits_{x\to2}\frac{(x-1)(x-2)}{(x-2)(x+2)}=\lim\limits_{x\to2}\frac{x-1}{x+2}=\frac{1}{4}$

Ex 18. $\lim\limits_{x\to1^+}\frac{x-1}{|x-1|}=\lim\limits_{x\to1^+}\frac{x-1}{x-1}=1\neq-1=\lim\limits_{x\to1^-}\frac{x-1}{-(x-1)}=\lim\limits_{x\to1^-}\frac{x-1}{|x-1|}$

Ex 19. $\lim\limits_{x\to1}\frac{\sqrt{x}-1}{x-1}=\lim\limits_{x\to1}\frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim\limits_{x\to1}\frac{1}{\sqrt{x}+1}=\frac{1}{\lim\limits_{x\to1}\sqrt{x}+1}=\frac{1}{1+\sqrt{\lim\limits_{x\to1}x}}=\frac{1}{2}$

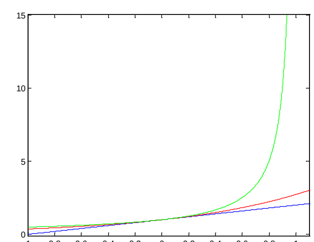
Ex 20. Let $f(x)=\begin{cases}3-x^2 & x\in Q\\ x & x\notin Q\end{cases}$, find c such that $\lim\limits_{x\to c}f(x)$ is exists.

Solution: Actually we need c such that $\lim\limits_{x\to c}3-x^2=3-c^2=c=\lim\limits_{x\to c}x$, thus

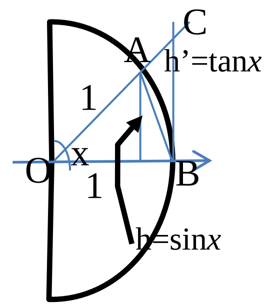
$3-c^2=c\Rightarrow c^2+c-3=0\Rightarrow c_{1,2}=\frac{-1\pm\sqrt{1+12}}{2}=\frac{-1\pm\sqrt{13}}{2}$

Theorem: If $f(x)\le g(x)$ in an open interval $x\in(a-b,a+b)$, then $\lim\limits_{x\to a}f(x)\le\lim\limits_{x\to a}g(x)$

The Squeeze Theorem (Other names: Sandwich Thrm, two policemen and a drunk theorem): If $f(x)\le g(x)\le h(x)$ in an open interval $x\in(a-b,a+b)$, and $\lim\limits_{x\to a}f(x)=L=\lim\limits_{x\to a}g(x)$, then $\lim\limits_{x\to a}h(x)=L$.



Ex 21. Let see that $\lim\limits_{x\to0}\frac{e^{ax}-1}{x}=a$. Since at $0\le ax<1$ we have $1+ax\le e^{ax}\le\frac{1}{1-ax}$ (see the graph), the squeeze theorem provides (in each part of inequality sub 1 and divide by x): $a\le\frac{e^{ax}-1}{x}\le\frac{1}{x}\left(\frac{1}{1-ax}-1\right)=\frac{a\not{x}}{\not{x}(1-ax)}\to a$



Ex 22. Let see why $\lim\limits_{x\to0}\frac{\sin x}{x}=1$: Let see the areas of "slice of pie" AOB and triangles AOB and COB: $S_{\pi AOB}=\frac{x}{2\pi}\pi1^2=\frac{x}{2}$, $S_{\triangle AOB}=\frac{1}{2}\cdot OB\cdot h=\frac{\sin x}{2}$, $S_{\triangle COB}=\frac{1}{2}\cdot OB\cdot h'=\frac{\tan x}{2}$. Using the

relationship between sizes of these shapes, one write $\sin x < x < \tan x$, next divide all by sin*x* to get $1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x} \xrightarrow[x \to 0]{} 1$, by squeeze theorem we got that $\lim_{x\to 0} \frac{x}{\sin x} = 1$, and the last trick is $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{1}{\frac{x}{\sin x}} = \frac{1}{\lim_{x\to 0} \frac{x}{\sin x}} = 1$.