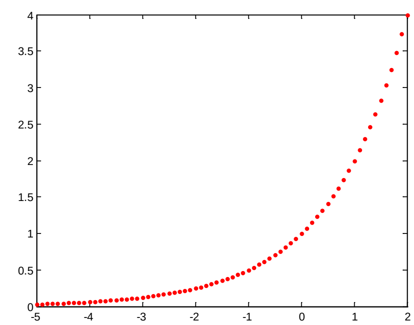
2 Exponential Functions (1.5)

Reminder: A function is said to be monotonically increasing if for all x, y such that $x \le y$ one has $f(x) \le f(y)$.

Reminder: In an expression b^e, ***b*** denoted as a base and ***e*** as an exponent. Consider two functions $f(x) = x^2$ and $g(x) = 2^x$. The first function, f, is said to be a **power function**, since the variable is the base. In the second function, g, the variable is an exponent; therefore we denote g as an **exponential function**. In general, for any (positive) constant a>0 the function $h(x) = a^x$ is

exponential function.



When x is an integer n>0 or a rational number q/p the expression a^x is clear: $a^n = \underbrace{a \cdot a \cdots a}_{n\,times}, a^{-n} = 1/a^n, a^{p/q} = \left(\sqrt[q]{a}\right)^p$

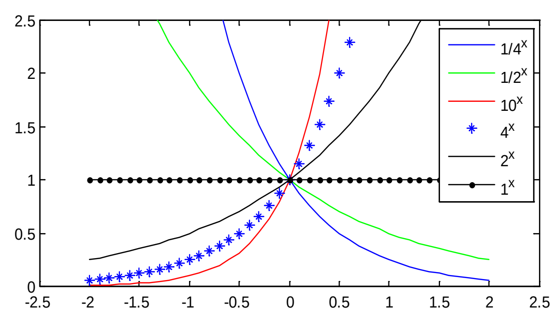
However, what does it mean when x is an irrational number, like 2^π ? Let's first sketch the function 2^x with a of real and rational values. For example some values

x	3	2	1	0	1/2	1/4	1/8
2^x	8	4	2	1	$\sqrt{2} \approx 1.41$	$2^{1/4} \approx 1.19$	$2^{1/8} \approx 1.09$

Our graph has "holes" at places of irrational numbers which we want to fill (we want to define 2^x there).

One can see that the function 2^x is monotonically increasing for real/rational numbers, therefore we want to define 2^x monotonically increasing on $\mathbb{R}$.

We know that $3.14 < \pi < 3.15$ ($\pi = 3.141592653589793$) therefore we must have $2^{3.14} < 2^\pi < 2^{3.15}$ (Note that decimal numbers with finite number of digits after period are rational). One can go for a better approximation of π, like $3.141 < \pi < 3.42$ or $3.1415 < \pi < 3.1416$ etc. It is provable that there is only one number greater then $2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots$ and less then $2^{3.15}, 2^{3.142}, 2^{3.1416}, \ldots$. Therefore we define 2^π to be this number, thus $2^\pi = 8.824977827076287$.



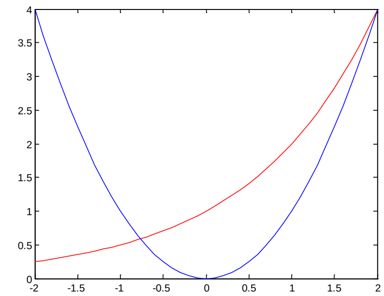
In a similar way one defines 2^x or a^x for $a > 0$. See the graph of exponential function for several values of a.

DEF: Laws of Exponent

$$a^{x+y} = a^x a^y \qquad a^{x-y} = a^x / a^y$$
$$\left(a^x\right)^y = a^{xy} \qquad \left(ab\right)^x = a^x b^x$$

Ex 1. $\frac{3^{-5}}{9^{-2}} = \frac{3^{-5}}{\left(3^2\right)^{-2}} = \frac{3^{-5}}{3^{-4}} = 3^{-1} = \frac{1}{3}$

Ex 2. $x\left(2x^4\right)^5 = x\left(2^5\left(x^4\right)^5\right) = x\left(2^5 x^{20}\right) = 2^5 x^{21}$



Ex 3. Sketch x^2 and 2^x to see differences.

Ex 4. The population of bacteria. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is p(t), where t is measured in hours, and the initial population is p(0)=1000, then we have

$$p(1) = 2p(0) = 2\cdot 1000$$
$$p(2) = 2p(1) = 2\cdot 2p(0) = 2^2\cdot 1000$$
$$\vdots$$
$$p(t) = 2^t\cdot 1000$$

2.1 The number *e* and the Natural Exponential Function

Every function a^x crosses the y-axis in the same place ($a^0 = 1$). However, the slope of the tangent line (it will be defined later in the course, consider it is a line that touch touches the graph only at this point) at point (0, 1) is different. It is convenient for future purposes that we use the function a^x which slope m=1 at (0, 1). Such number was found by Euler and is denoted as *e* a.k.a exponential. This is irrational number which value is:

$$e = 2.718281828459046.$$

We denote $f(x) = e^x$ the Natural Exponential Function.

Ex 5. The domain of $1/\left(e^{x^2-2}-1\right)$ is $\left\{x\in\mathbb{R} : x^2-2\neq 0\right\} = \mathbb{R}\setminus\left\{\pm\sqrt{2}\right\}$.