14.2 Series

Any rational number can be written as a finite sum of fractions: $0.123=\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}$.

A real number can also be written as sum of fractions, but when the number is irrational the sum will be infinite: $\pi=3+\frac{1}{10}+\frac{4}{10^2}+\frac{1}{10^3}+\frac{5}{10^4}+\frac{9}{10^5}+\frac{2}{10^6}+++$

Def: A sum of infinite sequence $\{a_n\}_{n=1}^{\infty}$, $S=\sum_{j=1}^{\infty}a_j$ is called **(infinite) series**.

Def: In order to find a sum of infinite sequence $S=\sum_{j=1}^{\infty}a_j$ one defines a sequence of **partial sums** as $\{S_n\}_{n=1}^{\infty}=\left\{\sum_{j=1}^{n}a_j\right\}_{n=1}^{\infty}$. If the sequence is convergent then $S=\lim_{n\to\infty}S_n$ and the series is called **convergent**. Otherwise the series is **divergent**.

Ex 1. $\sum_{k=1}^{\infty}k=\lim_{n\to\infty}\sum_{k=1}^{n}k=\lim_{n\to\infty}\frac{n(n-1)}{2}=\infty$

Ex 2. Compute $\sum_{k=1}^{\infty}\frac{1}{2^k}$. The partial sums are given as

$$\{S_n\}=\left\{\sum_{k=1}^{n}\frac{1}{2^k}\right\}=$$

$$\left\{\frac{1}{2},\frac{1}{2}+\frac{1}{4}=\frac{2+1}{4}=\frac{3}{4},\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{4+2+1}{8}=\frac{7}{8},\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{8+4+2+1}{16}=\frac{15}{16},\dots,\frac{2^n-1}{2^n}\right\}$$

so we need to computer $\lim_{n\to\infty}\frac{2^n-1}{2^n}=\lim_{n\to\infty}\frac{2^x-1}{2^x}\underset{L'Hospital}{=}\lim_{x\to\infty}\frac{2^x\ln 2}{2^x\ln 2}=1\Rightarrow\sum_{k=1}^{\infty}\frac{1}{2^k}=1$.

Ex 3. Compute $\sum_{k=1}^{\infty}\frac{1}{\sqrt{n^2+k}}$. The partial sums are given as $S_n=\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$, so we need to computer $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$.

Note, $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\neq\sum_{k=1}^{n}\lim_{n\to\infty}\frac{1}{\sqrt{n^2+k}}=0$ since n isn't finite.

$$1 \leftarrow \frac{n}{\sqrt{n^2+n}} = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}} \to 1 \Rightarrow \lim \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = 1$$

Ex 4. Geometric series: $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ... = \frac{a}{1-r}$ when $|r| < 1$, otherwise is divergent.

Ex 5. Telescopic series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ...\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Thm: If $S = \sum_{n=1}^{\infty} a_n$ is convergent series, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_{n+1} - S_n) = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = S - S = 0$$

The converse theorem doesn't true, see example below.

Ex 6. A hyper harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, e.g. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. For p=1, even though $\lim_{n\to\infty} \frac{1}{n} = 0$, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ is diverges since

$$k \ge n \Rightarrow \frac{1}{n} \ge \frac{1}{k} \Rightarrow S_k = \sum_{n=1}^{k} \frac{1}{n} \ge \sum_{n=1}^{k} \frac{1}{k} = \frac{1}{k} \sum_{n=1}^{k} 1 = \frac{1}{k}(1 + 2 + \cdots + k) = \frac{1}{k} \frac{k(k-1)}{2} = \frac{k-1}{2}$$

$$S = \lim_{k\to\infty} S_k \ge \lim_{k\to\infty} \frac{k-1}{2} = \infty$$

The test for Divergence: If $\lim_{n\to\infty} a_n \ne 0$ or DNE then $S = \sum_{n=1}^{\infty} a_n$ is divergent.

Ex 7. $\sum_{n=1}^{\infty} \frac{2^n}{3n+6}$ is diverges since $\lim_{n\to\infty} \frac{2^n}{3n+6} = \lim_{x\to\infty} \frac{2^x}{3x+6} = \lim_{x\to\infty} \frac{2^x \ln 2}{3} = \infty$

Thm: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (c is constant) and $\sum_{n=1}^{\infty} (a_n \pm b_n)$. Furthermore: $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.