13.3 Separable Equations (7.3)

Separable equation is an equation of the general form $\frac{dy}{dx}=f(x)g(y)$. When either g(y)=0 or f(x)=0 we get $\frac{dy}{dx}=0\Rightarrow y=const$, such solution called singular. Otherwise it can be rewritten as $\frac{dy}{g(y)}=f(x)dx$ and integrated, i.e. $\int\frac{dy}{g(y)}=\int f(x)dx$.

Ex 8. Solve $y'=xy$ with respect to $y(0)=1$. We use separation of variables

$$\frac{dy}{dx}=xy\Rightarrow\frac{dy}{y}=xdx\Rightarrow\ln|y|=\int\frac{dy}{y}=\int x\,dx=\frac{1}{2}x^2+C\Rightarrow|y|=e^{\ln|y|}=Ce^{\frac{1}{2}x^2}\Rightarrow y=ce^{\frac{1}{2}x^2}$$

Apply IC: $y(0)=ce^{\frac{1}{2}0^2}=c=1$ so $y=e^{\frac{1}{2}x^2}$. Verification: $y'=xe^{\frac{1}{2}x^2}=xy,\ y(0)=1$.

Ex 9. $y(1+x^2)dy+x(1+y^2)dx=0$

$$\begin{aligned}
&y(1+x^2)dy+x(1+y^2)dx=0\\
&y(1+x^2)dy=-x(1+y^2)dx\\
&\frac{y}{1+y^2}dy=-\frac{x}{1+x^2}dx\\
&\int\frac{y}{1+y^2}dy=-\int\frac{x}{1+x^2}dx\\
&\frac{1}{2}\ln\left(1+y^2\right)=-\frac{1}{2}\ln(1+x^2)+\frac{1}{2}\ln c\\
&\left(1+y^2\right)=\frac{c}{1+x^2}
\end{aligned}$$

Def: An orthogonal trajectory of a family of curves is a curve of the family orthogonally, that is, at right angles. For instance, y=mx family of lines\rays are orthogonal to $x^2+y^2=r^2$ family of circles.

Ex 10. Find the orthogonal trajectories of the $x=ky^2$ family of parabolas, where k is arbitrary constant.

We first need to find differential representation of the curve independent of the constant k. We first differentiate the equation then eliminate k using original equation, i.e. $k=\frac{x}{y^2}$. Thus: $1=2kyy'=2\frac{x}{y^2}yy'=2\frac{x}{y}y'\Rightarrow y'=\frac{y}{2x}$. The orthogonal trajectories slope has to be negative reciprocal $y'=-\frac{2x}{y}$.

We next solve this separable first order ODE

$$\frac{dy}{dx}=-\frac{2x}{y}\Rightarrow -ydy=2xdx\Rightarrow -\int y\,dy=\int 2x\,dx\Rightarrow -\frac{y^2}{2}=x^2+C$$

Thus the trajectories orthogonal to parabolas $x=ky^2$ are ellipses $x^2+\frac{y^2}{2}=C$.

Population growth(7.4-7.5):

Ex 11. Solve $\frac{dy}{dt}=ky$

$$\frac{dy}{y}=kdt\Rightarrow\int\frac{dy}{y}=\int k\,dt\Rightarrow \ln y=kt+\ln C\Rightarrow y=Ce^{kt}=y(0)e^{kt}$$

Ex 12. Solve $\frac{dy}{dt}=ky\left(1-\frac{y}{m}\right)$

$$\frac{dy}{y\left(1-\frac{y}{m}\right)}=kdt\Rightarrow\int\frac{dy}{y\left(1-\frac{y}{m}\right)}=\int k\,dt$$

$$tk-\ln c=\int\frac{mdy}{y(m-y)}=\int\frac{1}{y}+\frac{1}{m-y}dy=\ln y-\ln(m-y)$$

$$\frac{1}{c}e^{tk}=\frac{y}{m-y}\Rightarrow ce^{-tk}=\frac{m-y}{y}=\frac{m}{y}-1\Rightarrow 1+ce^{-tk}=\frac{m}{y}\Rightarrow y=\frac{m}{1+ce^{-tk}}$$

$$y(0)=\frac{m}{1+c}\Rightarrow c=\frac{m-y_0}{y_0}$$

Predator-Prey Systems (7.6):

A prey (rabbit) would populate exponentially as $r'=kr$ without a predator wolf. The population of wolfs would decline as $w'=-qw$ without the food, a rabbit. In this model, we consider that the rabbit is being eaten by a wolf at the rate proportional to the size of both populations. Similarly, the wolf survival rates depends on the food is proportional to *rw*. Thus the model is given as following system of equations:

$$\begin{cases} r' = \quad kr - arw \\ w' = -qw + brw \end{cases}$$

In order to solve system of equations one need to learn another course, but we still can use graphical method similar to **direction fields**, called **phase portrait**. The idea is to express one function of the system in terms of the other one.

$$\frac{dw}{dr} = \frac{dw}{dt}\frac{dt}{dr} = \frac{(br-q)w}{(k-aw)r}$$

One example of phase portrait of such system can be found in the book, but will look at simpler example below.

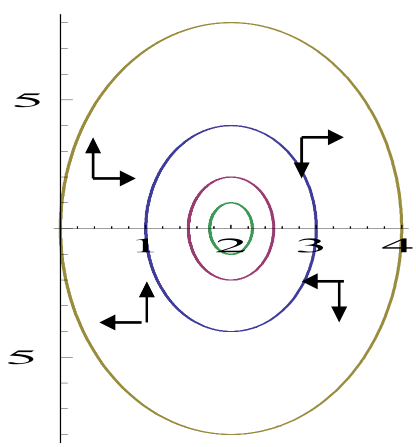
A system of equation can also arise from higher order DEs. More precisely any nth-order DE can be reformulated as a system of n one dimensional equations.

Ex 13. Convert $x'' = -4x + 8$ to system of equations and sketch its phase portrait.

$$\begin{cases} x' = y \\ y' = -4x + 8 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{-4x+8}{y}$$

The function y is increasing when $-4x + 8 > 0$, i.e. when $x<2$, has extreme value when $x=2$ and decreasing when $x>2$.

The function x is increasing when y>0, has extreme value at y=0 and decreasing when y<0.



$$ydy = (-4x+8)dx \Rightarrow \frac{y^2}{2} = -2x^2 + 8x$$

$$y^2 + 4x^2 - 16x = y^2 + 4(x-2)^2 - 16 = C$$

$$y^2 + 4(x-2)^2 = E \Rightarrow \frac{y^2}{E} + \frac{(x-2)^2}{E/4} = 1 \Rightarrow ellipse$$