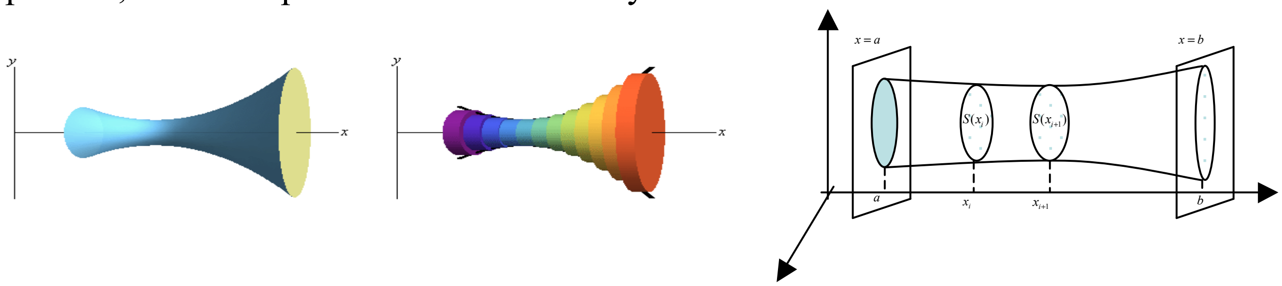
12.11.2 Volumes (6.2-6.3)

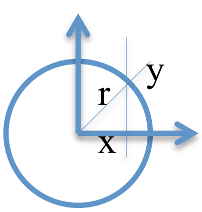
Volume of a a simple body like circular\elliptical cylinder or rectangular parallelepiped (a box) is given by V = Ah, where A is the area of the basis and h is the height. For a more complicated body B we cut it into small pieces and approximate it by cylinders and adding the volumes. The volume of B is obtained by limiting process, when the pieces become infinitely small.



We intersect\slicing B with a plane and obtaining a plane region that is called cross-sections of B. Let $A(x_i)$ be the area of cross section in a plane P_{x_i}. The volume of cylinder is $A(x_i^*)\Delta x$ and therefore the volume of the body is

$$V = \sum_{j=1}^{n} A(x_j^*)\Delta x = \int_a^b A(x)dx$$

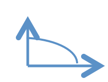
Ex 5. Volume of sphere



$$V = \int_{-r}^{r} A(x)dx = \int_{-r}^{r} \pi y^2\, dx = \int_{-r}^{r} \pi(r^2 - x^2)dx = 2\pi\int_0^r r^2 - x^2\, dx = 2\pi\left(r^2 x - \frac{x^3}{3}\right)_0^r =$$

$$= 2\pi\left(r^3 - \frac{r^3}{3}\right) = 2\pi r^3\left(1 - \frac{1}{3}\right) = 2\pi r^3 \cdot \frac{2}{3} = \frac{4}{3}\pi r^3$$

Ex 6. Find volume of body enclosed between the axes and $1 - x^3$ rotated around a) x-axis and b) y-axis.



When we rotate around x-axis, we “slice” into discs and so the volume is given by

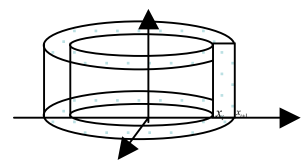
$V = \pi\int_a^b f^2(x)\,dx$ since the area is $\pi r^2 = \pi f(x)^2$, thus:

$$V_x = \pi\int_0^1 (1 - x^3)^2\, dx = \pi\int_0^1 1 - 2x^3 + x^6\, dx = \pi\left(x - \frac{2}{4}x^4 + \frac{1}{7}x^7\right)_0^1 = \pi\left(1 - \frac{1}{2} + \frac{1}{7}\right) = \frac{9}{14}\pi$$

In order to find the volume by on can "switch the axes" or in other words integrate by y, i.e. to find the area enclosed between axes and $x=\sqrt[3]{1-y}$. Thus

$$V_y=\pi\int_1^0\left(\sqrt[3]{1-y}\right)^2dy=\pi\int_1^0(1-y)^{2/3}dy=-\pi\int_1^0 t^{2/3}dt=\pi\int_0^1 t^{2/3}dt=\pi\left.\frac{t^{5/3}}{5/3}\right|_0^1=\frac{3}{5}\pi$$

12.11.2.1 Volumes by Cylindrical Shells (6.3)



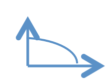
The other method to solve the last volume can be used for more complicated shapes is called "Method of Cylindrical Shells". Consider cylindrical shell with inner radius x_1 and outer radius x_2, the volume of the shell is the difference between the volumes of outer and inner cylinder. Thus

$$V=V_2-V_1=\pi x_2^2h-\pi x_1^2h=\pi\left(x_2^2-x_1^2\right)h=\pi\left(x_2+x_1\right)\left(x_2-x_1\right)h=$$
$$=2\pi\frac{x_2+x_1}{2}\left(x_2-x_1\right)h=2\pi r_{avg}(\Delta r)h=2\pi r_{avg}\cdot h\cdot(\Delta r)=\text{circumference}\cdot\text{height}\cdot\text{thikness}$$

When we get body obtained by rotating around y-axis, instead of slicing cross sections we divide\peel the body into cylindrical shells, such that *f*(*x*) is a height, the volume of every shell is given by $V_j=2\pi\overline{x}_j\cdot f\left(\overline{x}_j\right)\cdot\Delta x$ where $\overline{x}_j=\frac{x_j+x_{j+1}}{2}$ and therefore

$$V=\lim_{n\to\infty}\sum_{j=1}^{n}2\pi\overline{x}_jf\left(\overline{x}_j\right)\Delta x=\int_a^b 2\pi xf(x)dx$$

Ex 7. Find volume of body enclosed between the axes and $1-x^3$ rotated around y-axis using method of cylindrical shells.



$$V_y=2\pi\int_0^1 x\left(1-x^3\right)dx=2\pi\int_0^1 x-x^4\,dx=2\pi\left(\frac{x^2}{2}-\frac{x^5}{5}\right)_0^1=2\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3}{5}\pi$$

12.11.3 Arc Length (6.4)

When we speak about length of a curve we mean the distance between the end points of the graph after they are stretched like a graph was a string or thread. However we can't really do it this way, therefore we need to a better method.

We approximate the curve by straight lines, i.e. we divide the curve into small pieces each one approximated by a line. The length of the curve $y=f(x)$ is approximated by a sum of lengths of these lines.

Such lines can be formulated as

$$l_j(x)=\frac{x-x_{j-1}}{x_j-x_{j-1}}f(x_j)+\frac{x-x_j}{x_{j-1}-x_j}f(x_{j-1})=\frac{f(x_j)-f(x_{j-1})}{x_j-x_{j-1}}x+\frac{x_jf(x_{j-1})-x_{j-1}f(x_j)}{x_j-x_{j-1}}$$

The length of each line is given by (Pythagoras formula):

$$|l_j|=\sqrt{(x_j-x_{j-1})^2+(l_j(x_j)-l_j(x_{j-1}))^2}=\sqrt{(x_j-x_{j-1})^2+(f(x_j)-f(x_{j-1}))^2}=$$

$$=(x_j-x_{j-1})\sqrt{1+\left(\frac{f(x_j)-f(x_{j-1})}{x_j-x_{j-1}}\right)^2}\approx\sqrt{1+(f'(x))^2}\Delta x$$

Thus the length of the curve is given by

$$L=\lim_{n\to\infty}\sum_{j=1}^{n}|l_j|=\lim_{n\to\infty}\sum_{j=1}^{n}\sqrt{1+(f'(x))^2}\Delta x=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

Similarly, when the curve is given by $x=f(y)$ we have $L=\int_a^b\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$.

When the curve is given by parameterization $x=f(t),y=g(t)$ we have $f'(t_j)\approx\frac{\Delta x_j}{\Delta t}$, i.e. $\Delta x\approx f'(t)\Delta t$ and similarly $\Delta y\approx g'(t)\Delta t$, thus

$$|l_j|=\sqrt{(x_j-x_{j-1})^2+(y_j-y_{j-1})^2}\approx\sqrt{(f'(t)\Delta t)^2+(g'(t)\Delta t)^2}=\sqrt{(f'(t))^2+(g'(t))^2}\Delta t$$

and therefore $L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Consider a **polar curve** $r=f(\theta)$, in polar coordinates we have

$x=r\cos\theta=f(\theta)\cos\theta, y=r\sin\theta=f(\theta)\sin\theta$ and therefore $\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta$ and $\frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta$. Thus

$$L=\int_a^b\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_a^b\sqrt{(f'(\theta)\cos\theta-f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta+f(\theta)\cos\theta)^2}\,d\theta$$

$$=\int_a^b\sqrt{(f'(\theta))^2+f^2(\theta)}\,d\theta=\int_a^b\sqrt{r^2+(f'(\theta))^2}\,d\theta$$

Ex 8. Compute arc length for curve described by $(x,y)=(t,t^2)$ on $[0,1]$

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^1 \sqrt{1+(2t)^2}\,dt \underset{\substack{u=2t\\ du=2dt}}{=} \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du=$$

$$=u\sqrt{1+u^2}\Big|_0^2-\int_0^2 u\frac{2u}{2\sqrt{1+u^2}}du=\ldots=\frac{1}{2}\left(u\sqrt{1+u^2}+\ln\left(u+\sqrt{1+u^2}\right)\right)\Big|_0^2=\frac{1}{2}\left(2\sqrt{5}+\ln\left(2+\sqrt{5}\right)\right)$$

Ex 9. Compute arc length of $r=3\cos\theta+\sqrt{7}\sin\theta$

$$L=\int_a^b \sqrt{r^2+\left(f'(\theta)\right)^2}\,d\theta=\int_a^b \sqrt{\left(3\cos\theta+\sqrt{7}\sin\theta\right)^2+\left(-3\sin\theta+\sqrt{7}\cos\theta\right)^2}\,d\theta=\int_a^b \sqrt{9+7}\,d\theta=4(b-a)$$

12.11.4 Average value of function

Consider an average

$$Avg=\frac{y_1+y_2+\cdots+y_n}{n}=\frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{n}=\frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{\frac{b-a}{\Delta x}}=$$

$$=\frac{1}{b-a}\sum_{j=1}^{n} f\left(x_j\right)\Delta x=\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex 1. An experiment measurement showed that the investigated phenomena could be described as $f(x)=x^3-1$ on time interval $[1,3]$. Find the average value of this function.

$$Avg=\frac{1}{3-1}\int_1^3 x^3-1\,dx=\frac{1}{2}\left(\frac{x^4}{4}-x\right)\Bigg|_1^3=\frac{1}{2}\left(\frac{81}{4}-3-\frac{1}{4}+1\right)=\frac{81-9}{8}=\frac{72}{8}=9$$

Note that $x^3-1=9\Rightarrow 10=x^3\Rightarrow x=\sqrt[3]{10}$, i.e. $f(-2)=f_{avg}$. It is no coincidence.

Theorem: If *f*(*x*) is continues on [a,b], the there is exists a c in [a,b] such that:

$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$, in other words we have $\int_a^b f(x)\,dx=f(c)(b-a)$.