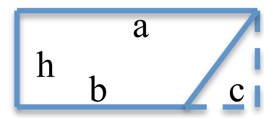
12.11 Application of Integration

12.11.1 Area between curves

Consider that we interesting to compute an area between two curves. How can we do that? The idea is similar to the one used to find the area of trapeze. For a example $ha-\frac{1}{2}hc=h(b+c)-\frac{1}{2}hc=hb+\frac{1}{2}hc=h\frac{a+b}{2}$



Let's rewrite this example so the area will be between 2 curves

$y_1=h$ and $y_2=\begin{cases}\frac{x-b}{a-b}h & x\ge b\\ 0 & else\end{cases}$, thus

$$\int_0^a y_1-y_2\,dx=\int_0^a h\,dx-\int_b^a\frac{x-b}{a-b}h\,dx=hx\Big|_0^a-\frac{h}{2}\frac{(x-b)^2}{a-b}\Bigg|_b^a=ha-\frac{a-b}{2}h=h\frac{a+b}{2}$$

Thm: The area of region bounded by continuous curves y=f(x) and y=g(x) is given by and the lines x=a and x=b, where f(x)>g(x) for all x in [a,b] is given by

$$A=\int_a^b f(x)-g(x)dx$$

If f(x) and g(x) are velocities of moving objects and f(a)=g(a) the area is interpreted as distance between the objects at time x=b.

Ex 1. To find the area enclosed between parabolas $5-x^2$ and x^2-3 we first need to find the points of intersection, which will be the limits of the integral. Thus: $5-x^2=x^2-3\Rightarrow 2x^2=5+3=8\Rightarrow x^2=4\Rightarrow x=\pm 2$ and so

$$A=\int_{-2}^{2}\left(5-x^2\right)-\left(x^2-3\right)dx=2\int_{-2}^{2}4-x^2\,dx=$$

$$2\left(4x-\frac{x^3}{3}\right)_{-2}^{2}=2\left(\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right)=2\left(16-\frac{16}{3}\right)=\frac{64}{3}$$

Ex 2. Draw curves $y^2=5-x$ and $y^2=3+x$. How to find the areas enclosed between them? It is possible to split it into 2 subdomains and sum their areas. The other way is to see that this is the same graph as in previous example (except that x and y were switched) and to integrate with respect to y.

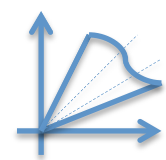
Ex 3. Consider we compute the area under a parametric curve given by $x = r\cos t, \quad y = r\sin t$ from x=-r to x=r:

$$\int_{-r}^{r} y(x)\,dx \underset{\substack{x = r\cos t \\ dx = x'(t)dt = -r\sin t\,dt}}{=} \int_{\pi}^{0} y(t)x'(t)\,dt = \int_{\pi}^{0} r\sin t(-r\sin t)\,dt = -r^2\int_{\pi}^{0}\sin^2 t\,dt =$$

$$= -\frac{r^2}{2}\int_{\pi}^{0}(1-\cos 2t)\,dt = -r^2\frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)_{\pi}^{0} = \frac{r^2\pi}{2}$$

12.11.1.1 Area in Polar coordinates (appendix H)

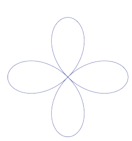
Polar coordinates are given by $x = r\cos\theta, \quad y = r\sin\theta \Rightarrow r^2 = x^2 + y^2 \quad \tan\theta = \dfrac{y}{x}$. A straight line (a ray) that passes through origin and making angle θ_0 with polar axis is given by formula $\theta = \theta_0$.



The rest of Appendix H.1 is for self-reading.

The area enclosed between θ_1 and θ_2 and constant radius *r* is given by $A = \frac{1}{2}r^2(\theta_2 - \theta_1)$. The area enclosed between θ_1 and θ_2 and a function $r = f(\theta)$ in polar coordinates is given by $A = \int_{\theta_1}^{\theta_2}\frac{1}{2}r^2\,d\theta = \int_{\theta_1}^{\theta_2}\frac{1}{2}f^2(\theta)\,d\theta = \sum_{j=1}^{n}\frac{1}{2}f^2(\theta_j^*)\Delta\theta$

Ex 4. Find area enclosed by one loop of the four leaved rose $r = \cos^2 2\theta$



$$A = \int_{-\pi/4}^{\pi/4}\frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\frac{1}{2}(1+\cos 4\theta)\,d\theta =$$

$$= \frac{1}{2}\int_{0}^{\pi/4}(1+\cos 4\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)_{0}^{\pi/4} = \frac{\pi}{8}$$