12.10 Improper Integrals (5.10)

Def: Consider that *f*(*x*) has a discontinuity $c\in[a,b]$, but continues at $c\neq x\in[a,b]$.
Including the case when c is an end point, i.e. $(c=a,b],[a,c=b)$. Then $\int_a^b f(x)dx$ called improper and understood as $\int_a^b f(x)dx=\lim_{x\to c}\int_a^c f(x)dx+\lim_{x\to c}\int_c^b f(x)dx$.

Ex 1. $$\int_0^2\frac{dx}{x-2}=\lim_{\lambda\to2^-}\int_0^\lambda\frac{dx}{x-2}=\lim_{\lambda\to2^-}\left[\ln|x-2|\right]_0^\lambda=-\ln2+\lim_{\lambda\to2^-}\ln(2-\lambda)=-\infty$$

Ex 2. $$\int_1^2\frac{dx}{\sqrt{4-x^2}}=\lim_{\lambda\to2^-}\int_1^\lambda\frac{dx}{\sqrt{4-x^2}}=\lim_{\lambda\to2^-}\left[\arcsin\frac{x}{2}\right]_1^\lambda=-\arcsin\frac{1}{2}+\lim_{\lambda\to2^-}\arcsin\frac{\lambda}{2}=-\frac{\pi}{6}+\frac{\pi}{2}=\frac{\pi}{3}$$

Ex 3.
$$\int_0^1 x\ln x\,dx=\lim_{\lambda\to0^+}\int_\lambda^1 x\ln x\,dx=\lim_{\lambda\to0^+}\left[\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right]_\lambda^1=\lim_{\lambda\to0^+}\left(-\frac{1}{4}-\frac{1}{2}\lambda^2\ln\lambda+\frac{1}{4}\lambda^2\right)=-\frac{1}{4}$$
$$\lim_{\lambda\to0^+}\lambda^2\ln\lambda=\lim_{\lambda\to0^+}\frac{\ln\lambda}{\lambda^{-2}}\underset{\substack{\infty/\infty\\ L'hospital}}{=}\lim_{\lambda\to0^+}\frac{1/\lambda}{-2\lambda^{-3}}=\lim_{\lambda\to0^+}\frac{\lambda^2}{-2}=0$$

Ex 4.
$$\begin{aligned}\int_{-1}^2 x^2\ln|x|\,dx &=\lim_{\lambda\to0^-}\int_{-1}^\lambda x^2\ln|x|\,dx+\lim_{\gamma\to0^+}\int_\gamma^2 x^2\ln|x|\,dx=\\ &=\lim_{\lambda\to0^-}\int_{-1}^\lambda x^2\ln(-x)dx+\lim_{\gamma\to0^+}\int_\lambda^2 x^2\ln x\,dx=\\ &=\lim_{\lambda\to0^-}\left[\frac{1}{3}x^3\ln(-x)-\frac{1}{9}x^3\right]_{-1}^\lambda+\lim_{\gamma\to0^+}\left[\frac{1}{3}x^3\ln(x)-\frac{1}{9}x^3\right]_\lambda^2\\ &=\lim_{\lambda\to0^-}\left(\frac{1}{3}\lambda^3\ln(-\lambda)-\frac{1}{9}\lambda^3-\frac{1}{9}\right)+\lim_{\gamma\to0^+}\left(\frac{8}{3}\ln2-\frac{8}{9}-\frac{1}{3}\gamma^3\ln\gamma+\frac{1}{9}\gamma^3\right)=\end{aligned}$$
$$=-\frac{1}{9}+\frac{8}{3}\ln2-\frac{8}{9}=-1+\frac{8}{3}\ln2$$

Another form of improper integral is an integral with infinite limits. It is useful for example for calculating probability. The probability that something will 100% happen after enough time has the form of $\int_0^\infty f(t)dt=1$.

Def: 1) If $\int_a^t f(x)dx$ for $t\geq a$ then $\int_a^\infty f(x)dx=\lim_{t\to\infty}\int_a^t f(x)dx$ provided the limit exists.

2) If $\int_t^a f(x)dx$ for every $t \le a$ then $\int_{-\infty}^a f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx$ provided the limit exists.

3) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ convergent then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$

Ex 5. $\int_1^{+\infty}\frac{dx}{x^2} = \lim_{\lambda\to+\infty}\int_1^\lambda \frac{dx}{x^2} = \lim_{\lambda\to+\infty}\left[\frac{-1}{x}\right]_1^\lambda = \lim_{\lambda\to+\infty}\frac{-1}{\lambda}+1 = 1$

Ex 6. $\int_{-\infty}^0 e^x\,dx = \lim_{\lambda\to-\infty}\int_\lambda^0 e^x\,dx = \lim_{\lambda\to-\infty}\left[e^x\right]_\lambda^0 = \lim_{\lambda\to-\infty}(1-e^\lambda) = 1$

Ex 7. $\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \lim_{\lambda\to-\infty}\int_\lambda^0\frac{dx}{1+x^2} + \lim_{\mu\to+\infty}\int_0^\mu\frac{dx}{1+x^2} = -\lim_{\lambda\to-\infty}\arctan\lambda + \lim_{\mu\to+\infty}\arctan\mu = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

Ex 8. $\int_1^{+\infty}\frac{dx}{x} = \lim_{\lambda\to+\infty}\int_1^\lambda\frac{dx}{x} = \lim_{\lambda\to+\infty}\left[\ln x\right]_1^\lambda = \lim_{\lambda\to+\infty}\ln\lambda - \ln 1 = +\infty$

Ex 9. $\int_0^1\frac{dx}{1-x} = \lim_{\lambda\to1^-}\int_0^\lambda\frac{dx}{1-x} = \lim_{\lambda\to1^-}\left[-\ln(1-x)\right]_0^\lambda = -\lim_{\lambda\to1^-}\ln(1-\lambda) = +\infty$

Ex 10. $\nexists\int_0^{+\infty}\sin x\,dx = \lim_{\lambda\to\infty}\int_0^\lambda \sin x\,dx = \lim_{\lambda\to\infty}\left(-\cos x\right)_0^\lambda = \nexists\lim_{\lambda\to\infty}\left(1-\cos\lambda\right)$

Ex 11. $\int_1^{+\infty}\frac{dx}{x^p} = \lim_{\lambda\to\infty}\int_1^\lambda\frac{dx}{x^p} = \lim_{\lambda\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^\lambda = \lim_{\lambda\to\infty}\frac{\lambda^{1-p}}{1-p} - \frac{1}{1-p} \underset{\lim_{\lambda\to\infty}\lambda^{1-p}}{=} \begin{cases} -\frac{1}{1-p} & p>1 \\ +\infty & p\le 1\end{cases}$

Ex 12. $\int_2^{+\infty}\frac{dx}{x\ln x\cdot\ln(\ln x)} \underset{\substack{x>1\Rightarrow \\ x>0,\ln x>0}}{=} \lim_{\lambda\to\infty}\int_2^\lambda\frac{\frac{1}{x\ln x}}{\ln(\ln x)}dx \underset{\substack{t=\ln\ln x \\ dt=1/(x\ln x)}}{=} \lim_{\lambda\to\infty}\int_2^\lambda\frac{dt}{t} = \ln|t|_2^\infty \underset{\ln\ln\ln x}{=} \infty$

Ex 13. $\int_1^{+\infty}\frac{dx}{1+x^2}$ converges because $\int_1^{+\infty}\frac{dx}{1+x^2} < \int_1^{+\infty}\frac{dx}{x^2}$ and $\int_1^{+\infty}\frac{dx}{x^2}$ converges

Ex 14. $\int_1^{+\infty}\frac{dx}{\sqrt{x^2+1}}$ diverges because $\int_1^{+\infty}\frac{dx}{x}$ diverges and also

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{x\to+\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty}\frac{1}{\sqrt{1+1/x^2}} = 1$$