1 Functions (1.3)

Reminder: **y =f(x)** mean that a function **f** uses a variable (an ingredient) **x** to make the result **y**.

1.1 Transformation of functions

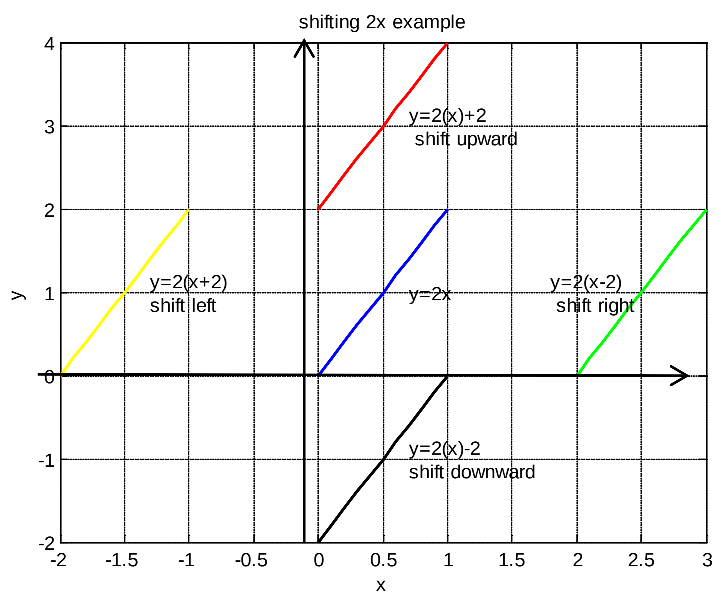
We know many elementary functions like $f(x)=x^3+2x^2+3x+1$, sin, cos or $\sqrt{\quad}$. One can obtain new functions by certain changes to the elementary functions. We denote these changes as transformations. By applying certain transformations to the graph of a given function one obtains the graphs of certain related functions, which helps to sketch them quickly by hand.

1.1.1 Shifting

The first sort of transformations we learn is shifting**.** We start a simple example followed by a more general formula and additional examples.

Ex 1. Let **y=2x** for $x\in[0,1]$.
Sketch it and sketch ot's shift right, left, upward and downward by 2 units.

Solution: (see the figure)



We next want **2x** be any function **f** and **2** be any (positive) constant **c>0**. Thus we define shifts as:

Shift		description
upward	y = f(x)+c	evaluate f at x and then add c
downward	y = f(x)-c	evaluate f at x and then subtract c
right	y = f(x-c)	evaluate f at x-c
left	y = f(x+c)	evaluate f at x+c

Ex 2. (1.3 q2a.q2b) Explain how each graph is obtained from the graph of y=f(x): a) y=f(x) + 8, b) y=f(x + 8).

Solution: a) the function is shifted 8 units up; b) the function is shifted 8 units left.

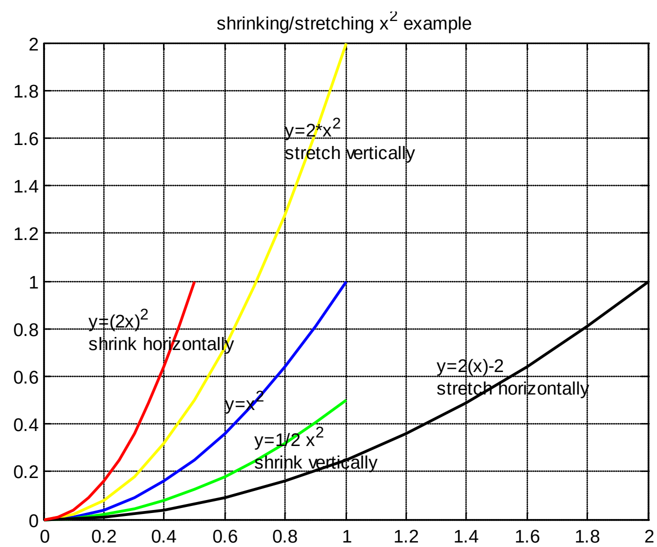
Ex 3. Sketch the functions from previous question given $f(x)=\sqrt{x}$

Ex 4. Shift **sin(x)** by 90 degrees compare with **cos(x).**

Solution: They are the same, remember an identity $\sin(x+90)=\cos(x)$

1.1.2 Shrinking and Stretching

We now provide an example of shrinking and stretching transformation.



Ex 5. Let $y=x^2$ for $x\in[0,1]$. Stretch and shrink it horizontally and vertically by factor 2.

Solution: (see the figure)

We next want x^2 be any function **f** and **2** be a constant **c>1**. Thus we define shrinks/stretches as:

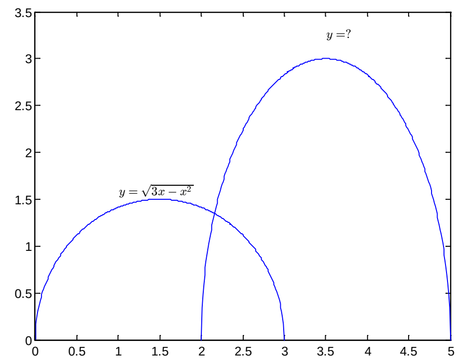
transformation		description
Shrink vertically	y = 1/c f(x)	evaluate f at x and then divide by c
Stretch vertically	y=c f(x)	evaluate f at x and then multiply by c
Shrink horizontally	y=f(c x)	evaluate f at cx
Stretch horizontally	y=f(x/c)	evaluate f at x/c

Ex 6. (1.3 q2a.q2b) Explain how each graph is obtained from the graph of y=f(x): a) y=8f(x), b) y=f(8x).

Ex 7. Sketch the functions from previous question given $f(x)=x^2$

Ex 8. Sketch **sin(nx) for n=1,2,3** and see what happened to the period**.**

Ex 9. (1.3 q6) The graph of $y=\sqrt{3x-x^2}$ is given. Use transformations to create the other function on the graph.

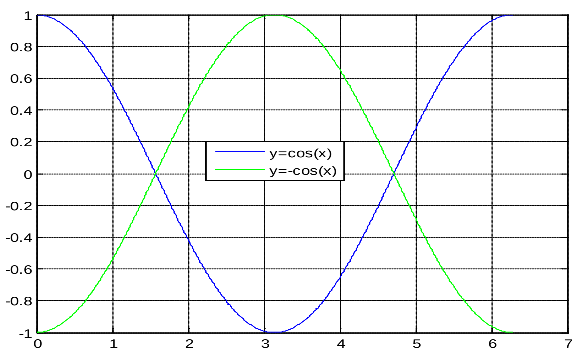


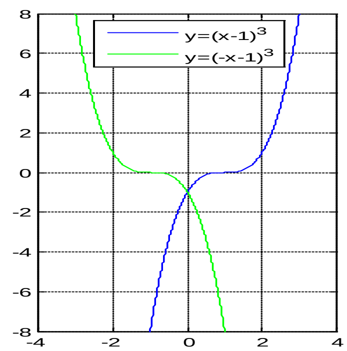
Solution: $y=2\sqrt{3(x-2)-(x-2)^2}$

1.1.3 Reflection

The last transformation we present is reflection about axes.

Ex 10. Reflect y=cos(x) and $y=(x-1)^3$ for $x\in\mathbb{R}$ about x-axis and about y-axis.





We finally generalize it for any function f and define

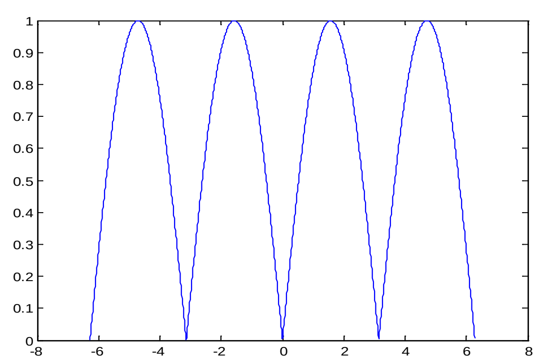
Reflect about		description
x-axis	y = -f(x)	evaluate f at x then change the sign
y-axis	y = f(-x)	evaluate f at -x

Ex 11. Sketch $y=x^2\pm 6x+10$

Solution: Note that $x^2\pm 6x+10=\left(x\pm 3\right)^2+1$, therefore the graphs are the graph of parabola shifted left\right by 3 units and shifter by 1 unit up.

Ex 12. Sketch $y=|\sin x|$

Solution: An absolute value of a function is not exactly a transformation (despite the fact that the book says it is), however



the knowledge of transformations helps to sketch such functions. The trick is to reflect the negative part of the function about x-axis while the positive part doesn't change.

1.2 Combinations of functions

Let f:A→R and g:B→R be real valued functions defined on domains A and B respectively. One may form new functions f+g, f-g, fg, f/g similarly to addition, subtraction, multiplication and division of real numbers.

The addition and subtraction is defined by $(f \pm g)(x) = f(x) \pm g(x)$. The domain of this new function is an intersection of the domains of f and g, i.e. $A \cap B$ because both f and g have to be defined.

Ex 13. Let $f(x) = \sqrt{x}$, which defined on $[0,\infty)$. Let $g(x) = \sin x$ (defined for all real numbers). Then $(f \pm g)(x) = \sqrt{x} \pm \sin x$ is defined on $[0,\infty)$.

The multiplication is similarly defined $(fg)(x) = f(x)g(x)$ and the domain is also $A \cap B$.

Ex 14. Let f, g be the functions defined in previous question, and then $fg(x) = \sqrt{x}\sin x$ is defined on $[0,\infty)$.

The division is little bit more involved since we can't divide by zero and therefore the domain of $(f/g)(x) = f(x)/g(x)$ must also exclude all zeros of $g(x)$, i.e. $\{x \in A \cap B : g(x) \neq 0\}$

Ex 15. Let f, g be the functions defined in previous question, and then $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sin x}$ is defined on $\{x \in \mathbb{R} : \sin x \neq 0\} = \{\mathbb{R} \ni x \neq k\pi : k \in Z\}$.

1.2.1 Composite Functions

The most interesting way to create new functions is function composition. Consider f:A→R and g:B→A, therefore $(f \circ g)(x) = f(g(x))$ is defined.

Ex 16. Let f, g be the functions defined in previous question, and then $(f \circ g)(x) = f(g(x)) = \sqrt{\sin(x)}$ isn't defined for all real numbers, but it can be defined on $\{x \in \mathbb{R} : \sin(x) \geq 0\}$, i.e. id I defined on

$...\cup[-2\pi,-\pi]\cup[0,\pi]\cup[2\pi,3\pi]\cup ... = \{x \in (2k\pi,(2k+1)\pi), k \in Z\}$

Ex 17. Let f, g be the functions defined in previous question, and then $(g \circ f)(x) = g(f(x)) = \sin\sqrt{x}$ is defined for all real numbers

Ex 18. Let $f(x) = x^2, g(x) = \sin x, h(x) = x + 2$ then

$f \circ g \circ h = f(g(h(x))) = f(g(x+2)) = f(\sin(x+2)) = \sin^2(x+2)$

Ex 19. (Decomposing a function) Let $F(x) = \cos((x+2)^2)$, find function f,g,h such that $F = f \circ g \circ h$.

Solution: $f(x) = \cos x, g(x) = x^2, h(x) = x + 2$