11.4 The l'Hospital's rule (4.5)

Cauchy's MVT: Let f, g be continuous on $\left[a,b\right]$ and differentiable on $\left(a,b\right)$, and assume $g'\left(x\right)\neq 0$. There is $c\in\left(a,b\right)$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

It is easy to see $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{b-a}\frac{b-a}{g(b)-g(a)}=\frac{f(b)-f(a)}{g(b)-g(a)}$, but why the same c for both?

Define $h(x)=f(x)+rg(x)$, where r is used to define h such that $h(a)=h(b)$, thus

$$h(a)=f(a)+rg(a)=f(b)+rg(b)=h(b)$$
$$\Rightarrow f(a)-f(b)=r\big(g(b)-g(a)\big)\Rightarrow r=-\frac{f(b)-f(a)}{g(b)-g(a)}$$

Now, by Rolle's theorem there is

$$0=h'(c)=f'(c)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(c)\Rightarrow\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Ex 10. Show that $1-\cos x<x^2/2,\ x\neq 0$ ➔ $\frac{\cos x}{1-x^2/2}=\frac{\sin c}{c}<1$ for some $c\in\left(0,x\right)$

The L'Hospital's Rule: The good use of Cauchy MVT is in the L'Hospital's (pronounced Loopeetal) rule, which say if $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the $\frac{"0"}{"0"},\frac{"\infty"}{"\infty"}$ form then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

If *f(x), g(x)* aren't defined at x=a, the discontinuity is removable. Therefore we can consider *f(a)=0=f(b)*.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-0}{g(x)-0}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-f(a)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{c\to a}\frac{f'(c)}{g'(c)}$$

Ex 1. $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$

Course: Accelerated Engineering Calculus I	Instructor: Michael Medvinsky

Ex 2. $\lim_{h\to 0}\frac{\sin h - h}{h^2} = \lim_{h\to 0}\frac{\cos h - 1}{2h} = \lim_{h\to 0}\frac{-\sin h}{2} = 0$

Ex 3. $\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{1/x}{-1/x^2} = \lim_{x\to 0}-\frac{x^2}{x} = \lim_{x\to 0} x = 0$

Ex 4. $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{\ln x^x} = \lim_{x\to 0} e^{x\ln x} = e^0 = 1$

Ex 5. $\lim_{x\to 0}\frac{e^{ax}-1}{x} = \lim_{x\to 0}\frac{ae^{ax}}{1} = a$

Ex 6. $\lim_{x\to 0}\frac{e^x - e^{-x}}{\ln(e-x)+x-1} = \lim_{x\to 0}\frac{e^x + e^{-x}}{-(e-x)^{-1}+1} = \frac{2}{-e^{-1}+1} = \frac{2e}{e-1}$

Ex 7. $\lim_{x\to 0}\frac{\sqrt{1+x}-\left(1+\frac{x}{2}-\frac{x^2}{8}\right)}{x^3} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}+\frac{x}{4}}{3x^2} = \lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}+\frac{1}{4}}{6x} = \lim_{x\to 0}\frac{\frac{3}{8}(1+x)^{-\frac{5}{2}}}{6} = \frac{1}{16}$

Ex 8. $\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} = \nexists \lim_{x\to 0}\frac{2x\sin\frac{1}{x}+\left(-\frac{1}{\cancel{x^2}}\right)\cancel{x^2}\cos\frac{1}{x}}{\cos x}$

but: $\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} = \lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0} x\sin\frac{1}{x} = 1\cdot 0 = 0$