11.3 Derivatives and the Shapes of Curves (4.3)

Rolle's Theorem: Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$ then there is $c\in(a,b)$ such that $f'(c)=0$.

Ex 6. If $f'(x)\neq 0,\forall x$ then there is no more than one solution to $f(x)=0$, since two solutions would imply, by Rolle's Theorem that there is some *x* s.t. $f'(x)=0$.

Ex 7. $f(x)=x^3+x\Rightarrow f'(x)=3x^2+1;\qquad x^3+x=x(x^2+1)=0\Leftrightarrow x=0$.

Ex 8. Show that there is exactly 2 solutions to $f(x)=x-\ln x-2=0$

For $x=e^{-2}$ we get $e^{-2}-\ln e^{-2}-2=e^{-2}+2-2=e^{-2}>0$, for $x=1:1-\ln 1-2=1-2=-1<0$ and for $x=e^2$: $e^2-\ln e^2-2=e^2-2-2=(e-2)(e+2)>0$. Thus we have at least 2 zeros, one in $[e^{-2},1]$ and one in $[1,e^2]$.

By Rolle 's Theorem there is a zero in the derivative between every pair of zeros in *f*, but $f'(x)=1-\dfrac{1}{x}$ have only one zero, therefore there is no more than 2 zeros to *f*.

La-grange's Mean Value Theorem (generalization of Rolle's Thm): Let f be continuous on $[a,b]$ and differentiable on (a,b). There is $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Def: Function *f* is increasing (decreasing) in interval *I* if $x_2>x_1\Rightarrow f(x_2)\geq f(x_1)$

($x_2>x_1\Rightarrow f(x_2)\leq f(x_1)$). If we change <,> with ≤,≥ then f is strictly icreasing/decreasing.

Increasing/Decreasing Test: If $f'(x)>0$ ($f'(x)<0$) on interval then f(x) is increasing (decreasing) on that interval.

One uses MVT to show that the sign of the derivative defines the increasing/decreasing behavior: Let *f'(x)*>0 on some interval *I*. Let $[x_1,x_2]\subset I$, so *f(x)* is differentiable there, and therefore by MVT three is exists $c\in[x_1,x_2]$ such that $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)>0$, but this is immediately imply $f(x_2)-f(x_1)>0$. Thus *f(x)* is increasing. Similarly, once can show *f'(x)*<0 implies *f* decreasing.

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The First Derivative Test: Suppose *c* critical number of continuous function *f*.

- If *f'* changes from positive to negative – *f(c)* is a local maximum
- If *f'* changes from negative to positive– *f(c)* is a local minimum
- If *f'* doesn't changes the sign then *f(c)* has no local max/min.

Concavity:

If *f'* is increasing (decreasing), a.k.a $f''(c)>0$ ($f''(c)<0$) function on an interval *I* then if concave upward (downward) aka smile(sad) on *I*. If f'' changes sign near *c* then there is an inflection point.

The Second Derivative Test:Suppose $f''(x)$ is continuous near *c*

- If $f'(x)=0$ and $f''(c)>0$ then f(c) is local minimum
- If $f'(x)=0$ and $f''(c)<0$ then f(c) is local maximum
- If $f'(x)=0$ and $f''(c)=0$ or doesn't exists – use another method (FDT)

Alg, Function investigation and Sketching Graph of function:

1. Find domain of definition.
2. Find whether the function is even $f(-x)=f(x)$, odd $f(-x)=-f(x)$ or neither.
3. Find the period p of a function $f(x+p)=f(x)$.
4. Find where the function crosses axes, aka solve $f(0), f(x)=0$
5. Find asymptotes, a line $l(x)$ if $\lim_{x\to\pm\infty}|f(x)-l(x)|=0$ or a line $x=x_0$ if $\lim_{x\to x_0}|f(x)|=\infty$
6. Do the following derivatives tests:
   a. Find critical points (end points and where $f'(x)=0$ or doesn't exists)
   b. Find where *f* is increasing $f'(x)>0$ or decreasing $f'(x)<0$.
   c. Define the concavity, $f''(x)>0$ smiling upward concavity, $f''(x)<0$ sad downward concavity.
   d. Classify extreme points Using First and/or Second Derivative tests.
7. Sketch the Graph

Ex 9. Sketch the graph of $f(x)=\frac{x^2+1}{2x}$

1. Domain of definition: $x\neq 0$

2. The function is odd: $f(-x)=\frac{(-x)^2+1}{-2x}=-\frac{x^2+1}{2x}=-f(x)$
3. The function isn't periodic.
4. $f(0)$ isn't defined and $f(x)\neq 0$
5. Asymptotes

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2+1}{2x^2}=\lim_{x\to\infty}\frac{1+1/x^2}{2}=\frac{1}{2}$$

$$b=\lim_{x\to\infty}f(x)-ax=\lim_{x\to\infty}\left(\frac{x^2+1}{2x}-\frac{1}{2}x\right)=\lim_{x\to\infty}\left(\frac{x^2+1-x^2}{2x}\right)=0$$

$$\lim_{x\to 0}\left|f(x)\right|=\lim_{x\to 0}\left|\frac{x^2+1}{2x}\right|=\infty$$

6. Do the following derivatives tests:
   a. Critical points (no end points) $f'(x)=\frac{4x^2-2\left(x^2+1\right)}{4x^2}=1-\frac{x^2+1}{2x^2}=\frac{1}{2}-\frac{1}{2x^2}$. $f'(x)=0$ if and only if $\frac{1}{2}=\frac{1}{2x^2}\Leftrightarrow x=\pm 1$
   b. Increasing at $\frac{1}{2}-\frac{1}{2x^2}<0\Rightarrow\frac{1}{2}<\frac{1}{2x^2}\Rightarrow x^2<1\Rightarrow -1<x<1$,

      Decreasing at $\frac{1}{2}-\frac{1}{2x^2}>0\Rightarrow\frac{1}{2}>\frac{1}{2x^2}\Rightarrow x^2>1\Rightarrow x>1\vee x<-1$.
   c. Concavity $f''(x)=\frac{1}{x^3}$, concave upward when x>0, downward when x<0
   d. $f''(x)=\left.\frac{1}{x^3}\right|_{\pm 1}=\pm 1$, thus we have local $\max f=-1,\min f=1$

Classify extreme points Using First and/or Second Derivative tests.

7. Sketch the Graph