Solving the Helmholtz Equation for General Geometry Using Simple Grids

M. Medvinsky∗ S. Tsynkov†‡ E. Turkel§

Abstract

The method of difference potentials was originally proposed by Ryaben’kii, and is a generalized discrete version of the method of Calderon’s operators. It handles non-conforming curvilinear boundaries, variable coefficients, and non-standard boundary conditions while keeping the complexity of the solver at the level of a finite-difference scheme on a regular structured grid. Compact finite difference schemes enable high order accuracy on small stencils and so require no additional boundary conditions beyond those needed for the differential equation itself. Previously, we have used difference potentials combined with compact schemes for solving transmission/scattering problems in regions of a simple shape. In this paper, we generalize our previous work to incorporate general shaped boundaries and interfaces.

Key words: Helmholtz equation, difference potentials, boundary projections, Calderon’s operators, regular grids, curvilinear boundaries, variable coefficients, compact differencing, high order accuracy

1 Introduction

We propose a high order accurate numerical method for the solution of two-dimensional boundary value problems of wave analysis. It applies to a wide variety of physical formulations that involve the transmission and scattering of acoustic and electromagnetic waves. In the current paper, we solve the scalar wave propagation problem in the frequency domain and concentrate on applications to general shaped boundaries and interfaces.

It is well known that high order accuracy is of central importance for the numerical simulation of waves because of the pollution effect [1, 2, 11]. For both finite difference and finite element approximations, the numerical phase velocity of the wave depends on the wavenumber \( k \), so a propagating packet of waves with different frequencies gets distorted in the simulation. Furthermore, the numerical error is proportional to \( h^p k^{p+1} \), where \( h \) is the grid size and \( p \) is the order of

∗Corresponding author. Department of Mathematics, University of Utah, Salt Lake City, Utah, USA. E-mail: medvinsky.michael@gmail.com
†Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695, USA. E-mail: tsynkov@math.ncsu.edu, URL: http://www.math.ncsu.edu/~tsynkov
‡Moscow Institute of Physics and Technology, Dolgoprudny, 141700, Russia.
§School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel. E-mail: turkel@post.tau.ac.il
accuracy of the chosen approximation. So the number of points per wavelength needed for a given 
accuracy increases as \( k^{1/p} \). Hence, higher order accurate approximations are very beneficial, as 
they reduce the effect of the pollution.

Our methodology for solving the Helmholtz equation combines compact equation-based 
high order accurate finite difference schemes [3, 4] with the method of difference potentials by 
Ryaben’kii [20, 21]. We have chosen this approach because we can take advantage of the simp-
licity and efficiency of high order finite difference schemes on regular structured grids (such as 
Cartesian or polar) and at the same time are able to handle non-conforming curvilinear boundaries 
and interfaces with no deterioration of accuracy due to staircasing.

The technique we propose presents a viable alternative to finite elements. Unlike finite dif-
fferences, finite elements are designed for handling sophisticated geometries. However, high order 
accurate finite element approximations can be built for arbitrary boundaries only in fairly sophis-
ticated and costly algorithms with isoparametric elements. These methods require grid generation 
which can be nontrivial for complex geometries and interfaces. In discontinuous Galerkin, dis-
continuous enrichment, and generalized finite element methods, high order accuracy also requires 
additional degrees of freedom. Yet, we are interested in predominantly smooth problems: geomet-
ically large regions with smooth material parameters separated by several interface boundaries, 
e.g., scatterers in large volumes of free space. The solution of such problems is smooth between 
the interfaces and so high order finite elements carry a substantial redundancy. The latter entails 
additional computational costs that we would like to avoid.

Therefore, we use compact finite difference schemes [3, 4, 9, 17, 25–27] to achieve high order 
accuracy. As any finite difference scheme, a compact scheme requires only one unknown per grid 
node, so there are no extra degrees of freedom. At the same time, unlike the standard high order 
accurate schemes compact schemes do not need extended stencils. In particular, equation-based 
compact schemes [3, 4] use the equation itself to eliminate the distant stencil points. These high 
order schemes reduce pollution while keeping the treatment of the boundary conditions simple, 
since the order of the resulting difference equation is equal to the order of the differential equation. 
Hence, no additional numerical boundary conditions are required.

The previous stages of development of our computational approach are reported in a series of 
papers [5, 6, 13, 14]. The method of difference potentials [20, 21] furnishes the required geometric 
flexibility. Specifically, it applies to a discretization on a regular structured grid and allows for non-
conforming curvilinear boundaries with no loss of accuracy. Our technique provides an attractive 
substitute for the method of boundary elements, because it is not limited to constant coefficients 
and does not involve singular integrals.

The method of difference potentials is a discrete analog of the method of Calderon’s operators 
[7, 21, 24]. It has the following key advantages:

- Maximum generality of boundary conditions. Any type of boundary conditions can be han-
dled with equal ease, including mixed, nonlocal and interfaces.

- The problem is discretized on a regular structured grid, yet boundaries and interfaces can 
have an arbitrary shape and need not conform to the grid. This causes no loss of accuracy 
due to staircasing.

- Variable coefficients, or equivalently, heterogeneous media, are easily handled. The con-
structs of Calderon’s operators remain essentially unchanged.
• The methodology does not require numerical approximation of singular integrals. The inverse operators used for computing the discrete counterparts to Calderon’s potentials and projections, involve no convolutions or singularities and allow fast numerical computation. The well-posedness of the discrete problem is guaranteed.

Our previous papers on the subject [5, 6, 13, 14] discussed model obstacles that were either circles or ellipses. The objective of this study is to include scatterers with more general shapes. The numerical results that we present demonstrate that this objective has been successfully achieved. Our algorithm attains the design fourth order accuracy when solving the transmission/scattering problems for a variety of non-conforming shapes, including the case of heterogeneous media.

1.1 Outline of the paper

In Section 2, we provide a brief account of the compact high order accurate equation-based schemes [3, 4, 9, 25] for solving the variable coefficient Helmholtz equation. In section 3, we introduce Calderon operators and their discrete counterparts. We briefly discuss their key properties. In Section 4 we discuss the coordinates associated with the interface curve and the equation-based extension. In section 5, we present the results of computations confirming the high order accuracy for non-conforming boundaries. Finally, section 6 contains conclusions.

2 Compact high order accurate equation-based schemes

We consider the Helmholtz equation in an unbounded heterogeneous medium that may have a discontinuity at an arbitrary shaped yet smooth interface between the subdomains $\Omega_0$ and $\Omega_1$:

$$L_1 u \overset{\text{def}}{=} \Delta u(x) + k_1^2(x) u = f_1(x), \quad x \in \Omega_1, \quad (1a)$$

$$L_0 u \overset{\text{def}}{=} \Delta u(x) + k_0^2 u = f_0(x), \quad x \in \Omega_0 = \mathbb{R}^n \setminus \Omega_1. \quad (1b)$$

The solution $u$ of equations (1) is assumed driven by the incident plane wave $u^{(\text{inc})} = e^{-ik_0(x \cos \theta + y \sin \theta)}$ and the source terms $f_j(x)$, $j \in \{0, 1\}$. Across the interface $\Gamma$ between $\Omega_1$ and $\Omega_0$, we require that the function $u$ and its first normal derivative be continuous. In addition, to ensure uniqueness, we impose the Sommerfeld radiation condition at infinity.

2.1 Scheme for the interior problem

For the interior problem formulated on $\Omega_1$, we consider the Helmholtz equation with a variable wavenumber (see equation (1a)):

$$L_1 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2(x, y) u = f(x, y). \quad (2)$$

We introduce a uniform in each direction Cartesian grid with the sizes $h_x$ and $h_y$ and use the following equation-based compact scheme:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} + (k^2 u)_{i,j}$$
+ \left( \frac{h_x^2}{12} + \frac{h_y^2}{12} \right) \frac{1}{h_x^2} \left( \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{h_y} \right) \\
-2 \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y} + \frac{u_{i-1,j+1} - 2u_{i-1,j} + u_{i-1,j-1}}{h_y} \\
+ \frac{h_x^2}{12} \left( \frac{k^2 u_{i+1,j} - 2(k^2 u)_{i,j} + (k^2 u)_{i-1,j}}{h_x^2} \right) \\
+ \frac{h_y^2}{12} \left( \frac{k^2 u_{i,j+1} - 2(k^2 u)_{i,j} + (k^2 u)_{i,j-1}}{h_y^2} \right) \\
= \frac{h_x^2}{12} f_{i+1,j} - 2f_{i,j} + f_{i-1,j} + \frac{h_y^2}{12} f_{i,j+1} - 2f_{i,j} + f_{i,j-1} + f_{i,j}. 
\tag{3}

Details of the derivation and accuracy analysis can be found in [4,9,25,28]. The scheme yields fourth order accuracy for smooth solutions. Even higher accuracy can be achieved using the same compact stencils. In [28], we have constructed a sixth order accurate equation-based scheme for the Helmholtz equation with a variable wavenumber $k$. Unlike regular schemes, the compact scheme (3) employs two stencils. The nine-node $3 \times 3$ stencil $\{(i, j), (i \pm 1, j), (i, j \pm 1), (i \pm 1, j \pm 1)\}$ is used for the discrete solution $u_{i,j}$, and the five-node stencil $\{(i, j), (i \pm 1, j), (i, j \pm 1)\}$ is used for the source function $f_{i,j}$. Since the left-hand side stencil is $3 \times 3$, the compact scheme (3) does not require additional boundary conditions beyond those needed for the original differential equation. Dirichlet boundary conditions are straightforward to set; Neumann boundary conditions can also be included without expanding the stencil [25,28].

### 2.2 Scheme for the exterior problem

For the exterior problem formulated on $\Omega_0$, we consider the Helmholtz equation with a constant wavenumber (see equation (1b)):

$$L_0 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + k^2 u = f(r, \theta). \tag{4}$$

The equation-based compact scheme is built on a uniform polar grid with the sizes $h_r$ and $h_\theta$:

$$1 \frac{1}{r \frac{1}{r_m} h_r} \left( r_{m+1/2} \frac{u_{m+1,l} - u_{m,l}}{h_r} - r_{m-1/2} \frac{u_{m,l} - u_{m-1,l}}{h_r} \right) \\
+ \frac{1}{r^2} \frac{1}{h_\theta} \left( \frac{u_{m,l+1} - 2u_{m,l} + u_{m,l-1}}{h_\theta} \right) \\
- \frac{2}{r^2} \frac{1}{r_{m-1}} \left( \frac{u_{m,l+1} - 2u_{m,l} + u_{m,l-1}}{h_\theta} \right) \\
+ \frac{1}{12h_\theta^2} \left[ \frac{1}{r^2_{m+1}} \left( u_{m+1,l+1} - 2u_{m+1,l} + u_{m+1,l-1} \right) \right] \\
- \frac{1}{12r_m} \left[ \frac{1}{r^2_{m-1}} \left( u_{m-1,l+1} - 2u_{m-1,l} + u_{m-1,l-1} \right) \right] \\
- \frac{h_r^2}{12r_m} \left[ \frac{\partial f}{\partial r} \right]_{m,l} - \frac{k^2 u_{m+1,l} - u_{m-1,l}}{2h_r} - \frac{1}{2h_r h_\theta^2} \left( \frac{1}{r^2_{m+1}} \left( u_{m+1,l+1} - 2u_{m+1,l} + u_{m+1,l-1} \right) \right) \left( u_{m-1,l+1} - 2u_{m-1,l} + u_{m-1,l-1} \right) \\
- \frac{1}{r^2_{m-1}} \left( u_{m-1,l+1} - 2u_{m-1,l} + u_{m-1,l-1} \right). \tag{5}$$
\[- \frac{h_r^2}{12r_m^2} \left( f_{m,l} - k^2 u_{m,l} - \frac{1}{r_m^2 h_\theta^2} (u_{m,l+1} - 2u_{m,l} + u_{m,l-1}) \right) + \frac{h_r}{12r_m^3} (u_{m+1,l} - u_{m-1,l}) \]
\[- \frac{h_\theta^2}{12} \left[ \frac{\partial^2 f}{\partial \theta^2}_{m,l} - k^2 u_{m,l+1} - 2u_{m,l} + u_{m,l-1} \right] \]
\[+ \frac{1}{12h_\theta^2 r_m} \left( r_{m+1/2}(u_{m+1,l+1} - u_{m,l+1}) - r_{m-1/2}(u_{m,l+1} - u_{m-1,l+1}) \right) \]
\[- 2 \left( r_{m+1/2}(u_{m+1,l} - u_{m,l}) - r_{m-1/2}(u_{m,l} - u_{m-1,l}) \right) \]
\[+ r_{m+1/2}(u_{m+1,l-1} - u_{m,l-1}) - r_{m-1/2}(u_{m,l-1} - u_{m-1,l-1}) \]}
\[+ k^2 u_{m,l} = f_{m,l} \]

Scheme (5) yields fourth order accuracy for smooth solutions; it is analyzed and tested in [3]. Similarly to scheme (3), the compact scheme (5) also employs two stencils. The stencil for the discrete solution \( u_{m,l} \) is nine-node: \( \{(m,l), (m \pm 1, l), (m, l \pm 1), (m \pm 1, l \pm 1)\} \), and the stencil for the right-hand side \( f_{m,l} \) is five-node: \( \{(m,l), (m \pm 1, l), (m, l \pm 1)\} \). Scheme (5) requires no additional boundary conditions beyond those needed for the original differential equation, since the left-hand side stencil is \( 3 \times 3 \). Neumann boundary conditions for scheme eqrefeq:PolarSchm can be set using the same \( 3 \times 3 \) stencil, see [13].

3 Difference potentials and projections

3.1 Auxiliary problem

The original domains \( \Omega_1 \) and \( \Omega_0 \), see (1), may have a general irregular shape. This could have made setting the boundary conditions for schemes (3) and (5), respectively, difficult, given that both schemes are constructed on regular structured grids. The method of difference potentials, however, allows us to completely circumvent those difficulties.

We enclose each domain \( \Omega_j \), \( j \in \{0, 1\} \), within an auxiliary domain \( \tilde{\Omega}_j \): \( \Omega_j \subset \tilde{\Omega}_j \). On the domain \( \tilde{\Omega}_j \), we formulate a special auxiliary problem (AP) for the corresponding inhomogeneous Helmholtz equation. The key requirement of the AP is that it should have a unique solution for any right-hand side defined on \( \tilde{\Omega}_j \). It can otherwise be formulated arbitrarily except that the exterior AP needs to include an exact or approximate counterpart of the Sommerfeld radiation condition. Hence, we choose the AP so that it is be easy to solve numerically. In particular, we select the auxiliary domains to be of simple shape, rectangular for \( \tilde{\Omega}_1 \) and circular for \( \tilde{\Omega}_0 \). In the method of difference potentials, the AP is used for computing the discrete counterparts of Calderon’s operators [7, 24]. While the operators themselves depend on the choice of the AP, the actual solution \( u \) of the problem of interest, e.g., the interface problem (1), is not affected [21].
3.1.1 Interior AP

The interior AP is formulated using Cartesian coordinates and the rectangular domain \( \widetilde{\Omega}_1 = [x_0, x_1] \times [y_0, y_1] \):

\[
L_1 u = g_1, \quad x \in \widetilde{\Omega}_1, \\
u = 0, \quad y \in \{y_0, y_1\}, \\
\frac{\partial u}{\partial x} = iku, \quad x = x_0, \\
-\frac{\partial u}{\partial x} = iku, \quad x = x_1.
\] (6)

The AP (6) is approximated by means of the compact scheme (3) and then solved using separation of variables and the FFT. Complex Robin boundary conditions imposed at the left and right edges of the auxiliary domain \( \Omega_1 \) are not intended to represent any physical behavior. They merely make the spectrum of the AP complex and hence ensure the uniqueness of the solution. These boundary conditions can be approximated with fourth order accuracy without having to extend the compact stencil of the scheme (3), see [4, Section 4.2].

Let \( N_1 \) be a uniform Cartesian grid on the rectangle \( \widetilde{\Omega}_1 \):

\[
N_1 = \{(x_m, y_n) \mid x_m = mh, \ y_n = nh, \ m = 0, \ldots, M_1, \ n = 0, \ldots, N_1\}.
\]

Creating a discrete analog of the boundary curve \( \Gamma \) is central to our method since \( \Gamma \) is not aligned with the Cartesian grid. The following subsets of the Cartesian grid \( N_1 \) are used for this purpose. Let \( M_1 \subset N_1 \) be the set of only interior nodes of the square domain \( \Omega_1 \). Thus \( M_1 \) contains all the nodes of \( N_1 \) except for those along the boundary edges of the rectangle:

\[
M_1 = \{(x_m, y_n) \mid x_m = mh, \ y_n = nh, \ m = 1, \ldots, M_1 - 1, \ n = 1, \ldots, N_1 - 1\}.
\]

Notice, that if we form a set which contains all of the nodes “touched” by the 9-point compact stencil operating on the set \( M_1 \), then this set will coincide with \( N_1 \). It is for this reason that the right-hand side \( g_{m,n} \) of the discrete AP is defined only on the interior nodes, i.e., on \( M_1 \). We now distinguish those nodes which are inside the boundary curve \( \Gamma \) from those which are outside \( \Gamma \). Let all those nodes which are confined within the continuous boundary \( \Gamma \) be denoted by \( M_{1}^+ \subset M_1 \) and those which are outside, except for the edges of the auxiliary domain, by \( M_{1}^- \subset M_1 \). Next, let the collections of all nodes touched by the 9-point compact stencil operating on the nodes of \( M_{1}^+ \) and \( M_{1}^- \) be referred to as \( N_{1}^+ \) and \( N_{1}^- \), respectively. Then, there is a nonempty intersection between the sets \( N_{1}^+ \) and \( N_{1}^- \). This is referred to as the grid boundary:

\[
\gamma_1 = N_{1}^+ \cap N_{1}^-.
\] (7)

All the foregoing grid sets are shown in Figure 1.
3.1.2 Exterior AP

The domain of the exterior problem is $\Omega_0 = \mathbb{R}^n \setminus \Omega_1$. The corresponding auxiliary domain $\tilde{\Omega}_0$ is chosen to be an annulus $\{r_0 \leq r \equiv \|x\| \leq r_1\}$ that contains the interface curve $\Gamma$. Then, we formulate the exterior AP as follows:

$$
L_0 u = g_0, \quad r_0 < r = \|x\| < r_1,
$$
$$
u = 0, \quad r = r_0,
$$
$$
Tu = 0, \quad r = r_1.
$$

(8)

The operator boundary condition $Tu = 0$ in (8) is equivalent to the Sommerfeld radiation condition, but set at the finite boundary $r = r_1$ rather than at infinity. The operator $T$ can be explicitly constructed in the transformed space after the variables in problem (8) have been separated by means of the azimuthal Fourier transform. In doing so, the entire AP (8) is also solved by separation of variables, which leads to a very efficient numerical procedure, see [3, Section 4].

We denote by $N_0$ the uniform polar grid on the annulus $\tilde{\Omega}_0$: 

$$
N_0 = \{(r_m, \theta_n) \mid r_m = mh, \theta_n = nh, m = 0, \ldots, M_0, n = 0, \ldots, N_0\}.
$$

A discrete analog of the boundary curve $\Gamma$ for the polar grid is defined similarly to that for the Cartesian grid. Namely, let $M_0 \subset N_0$ be the set of only interior nodes of the polar domain $\tilde{\Omega}_0$:

$$
M_0 = \{(r_m, \theta_n) \mid r_m = mh, \theta_n = nh, m = 1, \ldots, M_0 - 1, n = 1, \ldots, N_0 - 1\}.
$$

Let all of those nodes which are confined within the continuous boundary $\Gamma$ be denoted by $M^+_0 \subset M_0$, and those which are outside, except for the edges of the auxiliary domain, by $M^-_0 \subset M_0$. Next, let the collections of all nodes touched by the 9-point compact stencil operating on the nodes of $M^+_0$ and $M^-_0$ be referred to as $N^+_0$ and $N^-_0$, respectively. The intersection of the sets $N^+_0$ and $N^-_0$ is nonempty. The grid boundary is given by

$$
\gamma_0 = N^+_0 \cap N^-_0.
$$

(9)
The polar grid sets are shown in Figure 2.

![Figure 2: Interior and exterior grid subsets and the grid boundary for the exterior sub-problem.](image)

3.1.3 $\gamma$ for general shapes

The key factor in obtaining $\gamma$ with the help of formulae (7) or (9) is the decision whether a given grid node is interior to the body enclosed by $\Gamma$ or exterior, i.e., to define the sets $M_j^+, M_j^-, j \in \{0, 1\}$, which, in turn, are used to obtain the sets $N_j^+, N_j^-$. Regardless of the choice of grid such a decision is simple for a circle with a given radius $R_0$ and centered at the origin, e.g. $M_j^+ = \{(x, y) | \sqrt{x^2 + y^2} \leq R_0\}$. It is similarly simple for ellipses of a given eccentricity $e_0$ and centered at the origin, e.g. $M_j^+ = \{(x, y) | \text{Re} \cosh(x + iy) \leq \cosh e_0^{-1}\}$.

We now describe the procedure for a general shaped boundary $\Gamma$. Let $R = (R_x(t), R_y(t))$ be the parameterization of a star-shaped interface/boundary $\Gamma$. Let $p = (x, y)$ be a grid point. In order to decide whether $p$ is exterior or interior with respect to $\Gamma$, one compares the magnitude of the vector from the origin to the point $p$ and the value of $R$ in the same direction/angle $\theta$. The curve $R$ is a known function of the parameter $t$ where in general $t \neq \theta$. To match between $t$ and $\theta$ one uses a root finding algorithm, e.g., Newton-Raphson, to solve

$$\cos \frac{y}{x} = \cos \frac{R_y(t)}{R_x(t)}$$

for $t$. Once the matching is found the point $p = (x, y)$ is interior if it satisfies

$$\sqrt{x^2 + y^2} < \sqrt{R_x^2(t(\theta)) + R_y^2(t(\theta))},$$

otherwise $p$ is exterior.
3.2 Difference potentials

We now provide a brief description of the difference potentials and projections, while referring the reader to [13–15, 21] for a comprehensive account of the methodology. In particular, the well-posedness is discussed in [13, Section 3.1.4] and in [21, Part I]. The accuracy was investigated by Reznik [19], and some results are outlined in [13, Section 4.4]. The complexity is analyzed in [13, Section 4.6]. Solutions of some exterior and interface problems are presented in [14]. Algorithms and examples can be found in [15].

Let \( \xi_{\gamma_1} \) be a function specified at the grid boundary \( \gamma_1 \) of (7). Let \( w \) be a grid function on \( \mathbb{N}_1 \) that satisfies the discretized boundary conditions of the interior AP (6) at \( \partial \Omega_1 \), and also \( w|_{\gamma} = \xi_\gamma \Leftrightarrow \mathbf{Tr}_\gamma^{(h)} w = \xi_\gamma \). The difference potential with density \( \xi_{\gamma_1} \) is defined as

\[
P_{\mathbb{N}_1^+} \xi_{\gamma_1} = w - G_1^{(h)} \left( L_1^{(h)} w|_{\mathbb{M}_1^+} \right), \quad n \in \mathbb{N}_1^+,
\]

where \( L_1^{(h)} \) is the discrete counterpart of the continuous operator \( L_1 \) of (6) or (2) and \( G_1^{(h)} \) is its inverse obtained by solving the interior difference AP of Section 3.1.1 on the grid \( \mathbb{N}_1 \). Accordingly, the difference boundary projection is given by \( P_{\gamma_1} \xi_{\gamma_1} = \mathbf{Tr}_\gamma^{(h)} P_{\mathbb{N}_1^+} \xi_{\gamma_1} \). The key property of the projection \( P_{\gamma_1} \) is that a given \( \xi_{\gamma_1} \) satisfies the difference boundary equation with projection (BEP):

\[
P_{\gamma_1} \xi_{\gamma_1} + \mathbf{Tr}_\gamma^{(h)} G_1^{(h)} f_1^{(h)} = \xi_{\gamma_1}
\]

iff there exists \( u \) on \( \mathbb{N}_1^+ \) that satisfies equation (3) on \( \mathbb{M}_1^+ \) and such that \( \mathbf{Tr}_\gamma^{(h)} u = \xi_{\gamma_1} \). Note, that \( f_1^{(h)} \) in formula (12) is the discrete source term of the Helmholtz equation after the application of the second, i.e., five-node, stencil of the compact scheme, see the right-hand side of equation (3).

The constructs of discrete operators for the exterior domain exploit the grid sets and the auxiliary problem introduced in Section 3.1.2. The potential on \( \mathbb{N}_0^- \) is given by

\[
P_{\mathbb{N}_0^-} \xi_{\gamma_0} = w - G_0^{(h)} \left( L_0^{(h)} w|_{\mathbb{M}_0^-} \right),
\]

and the projection on \( \gamma_0 \) is defined as \( P_{\gamma_0} \xi_{\gamma_0} = \mathbf{Tr}_\gamma^{(h)} P_{\mathbb{N}_0^-} \xi_{\gamma_0} \), so that the discrete exterior BEP becomes

\[
P_{\gamma_0} \xi_{\gamma_0} + \mathbf{Tr}_\gamma^{(h)} G_0^{(h)} f_0^{(h)} + (I - P_{\gamma_0}) \mathbf{Tr}_\gamma^{(h)} u^{(inc)} = \xi_{\gamma_0}.
\]

The method of difference potentials requires no approximation of the boundary or interface conditions on the grid, and avoids unwanted staircasing effects [8, 10]. Let \( \xi_\Gamma = (\xi_0, \xi_1)|_\Gamma \) be the unknown vector function defined at the continuous boundary \( \Gamma \). We think of it as of the trace of the solution \( u \) and its first normal derivative. Suppose \( \xi_\Gamma \) has an expansion with respect to some basis (Fourier or Chebyshev) chosen on \( \Gamma \):

\[
\xi_\Gamma = (\xi_0, \xi_1)|_\Gamma = \sum_{n=-M}^{M} c_n^{(0)}(\psi_n, 0) + \sum_{n=-M}^{M} c_n^{(1)}(0, \psi_n),
\]

where \( c_n^{(0)} \) and \( c_n^{(1)} \) are the coefficients to be determined. The summation in (15) can be taken finite because for sufficiently smooth \( \xi_\Gamma \) the corresponding Fourier or Chebyshev series converges
rapidly. Hence, even for relatively small $M$ the spectral representation (15) provides the accuracy beyond the one that can be obtained on the grid inside the computational domain.

Using Taylor’s formula with equation-based derivatives [6, 13, 19], we extend $\xi_\Gamma$ from $\Gamma$ to the nodes of $\gamma_1$ located nearby:

$$\xi_{\gamma_1} = E x^{(1)} \xi_\Gamma = E x_{H}^{(1)} (\xi_0, \xi_1) |_\Gamma + E x_{I}^{(1)} f_1.$$  \hspace{1cm} (16)

Similarly for the exterior part, we introduce another equation-based Taylor extension:

$$\xi_{\gamma_0} = E x^{(0)} \xi_\Gamma = E x_{H}^{(0)} (\xi_0, \xi_1) |_\Gamma + E x_{I}^{(0)} f_0.$$  \hspace{1cm} (17)

In formulae (16) and (17), the operators $E x_{H}^{(j)}$ and $E x_{I}^{(j)}$, $j = 0, 1$, denote the homogenous and inhomogeneous part of the overall extension, respectively. Taking $\xi_\Gamma$ in the form (15), we rewrite (16) and (17) as follows:

$$\xi_{\gamma_j} = E x^{(j)} \xi_\Gamma = E x^{(j)} \left( \sum_{n=-M}^{M} c_n^{(0)} (\psi_n, 0) + \sum_{n=-M}^{M} c_n^{(1)} (0, \psi_n) \right)$$

$$= \sum_{n=-M}^{M} c_n^{(0)} E x_{H}^{(j)} (\psi_n, 0) + \sum_{n=M}^{M} c_n^{(1)} E x_{H}^{(j)} (0, \psi_n) + E x_{I}^{(j)} f_j.$$  \hspace{1cm} (18)

We then substitute the extension (16) into the BEP (12) and the extension (17) into the BEP (14), which yields a system of linear equations to be solved with respect to the coefficients of (15):

$$\begin{bmatrix} Q^{(0,1)} & Q^{(1,1)} \\ Q^{(0,0)} & Q^{(1,0)} \end{bmatrix} \begin{bmatrix} c^{(0)} \\ c^{(1)} \end{bmatrix} = \begin{bmatrix} -Tr_{\gamma_1}^{(h)} G_1^{(h)} f_1^{(h)} - (P_{\gamma_1} - I) E x_{I}^{(1)} f_1 \\ -Tr_{\gamma_0}^{(h)} G_0^{(h)} f_0^{(h)} - (P_{\gamma_0} - I) (E x_{I}^{(0)} f_0 - Tr_{\gamma_0}^{(h)} u^{(inc)}) \end{bmatrix}.$$  \hspace{1cm} (19)

In formula (18),

$$c = [c^{(0)}, c^{(1)}]^T = [c^{(0)}_{-M}, \ldots, c^{(0)}_{M}, c^{(1)}_{-M}, \ldots, c^{(1)}_{M}]^T$$

and the columns of sub-matrices $Q^{(k,j)}$ and $Q^{(k,j)}$, $k = 0, 1$, $j = 0, 1$, are given by

$$Q^{(k,j)}_n = (P_{\gamma_j} - I) E x_{H}^{(j)} \psi_n^k,$$

where $\psi_0^n = (\psi_n, 0)$ and $\psi_1^n = (0, \psi_n)$, $n = -M, \ldots, M$.

We emphasize that since $\xi_\Gamma$ appears in both (16) and (17), which translates into the system (18) written with respect to the same set of coefficients (19), then the interface condition, that requires the continuity of the solution $u$ and it’s normal derivative across $\Gamma$ (see the beginning of Section 2), is automatically enforced.

System (18) is typically overdetermined. It is solved in the sense of least squares using a QR decomposition. We note that even though the number of equations in the system (18) exceeds the number of unknowns, its least squares solution is “almost classical” in the sense that the residual of (18) at the minimum is small and converges to zero as the grid size decreases. Finally, once $\xi_\Gamma$ has been obtained in the form (15), we apply the extension operators (16) and (17) once again and then compute the discrete interior and exterior solutions as the difference potentials (11) and (13), respectively.
4 Coordinates associated with a curve and equation-based extension

The extension operators (16) and (17) are of key significance for the application of the method of difference potentials. In this section, we construct these equation-based extensions on an arbitrarily shaped, yet smooth, simple closed interface curve \( \Gamma \). It is most natural to describe the extension in terms of the arc length parameterization of \( \Gamma \). On the other hand, for a generally shaped curve it is often impossible to analytically obtain its arc length parameterization. Therefore, in many cases, it might be convenient to employ a different parameterization than arc length. Then, one would use the chain rule to obtain the required normal derivatives.

4.1 Curvilinear coordinates

Assume that \( \Gamma \) is parameterized by its arc length \( s \):

\[
\Gamma = \{ R(s) = (R_x(s), R_y(s)) | 0 \leq s \leq S \},
\]

where \( R \) is the radius-vector that traces the curve. Assume, for definiteness, that as \( s \) increases the point \( R(s) \) moves counterclockwise along \( \Gamma \). The unit tangent (\( \tau \)) and the unit normal (\( \nu \)) vectors to \( \Gamma \) are defined as

\[
\tau = \tau(s) = \frac{dR}{ds} \quad \text{and} \quad \nu = (\nu_x, \nu_y) = (\tau_y, -\tau_x).
\] (20)

Given a counterclockwise parametrization \( R = R(s) \), the normal \( \nu \) always points outward with respect to the domain \( \Omega_1 \). Hence, the pair of vectors (\( \nu, \tau \)) always have a fixed right-handed orientation in the plane.

The relation between the tangent \( \tau \), the normal \( \nu \), and the curvature \( \zeta \) of the curve \( \Gamma \) is given by the Frenet formula:

\[
\frac{d\tau}{ds} = \zeta \nu.
\] (21)

The vector \( \frac{d\tau}{ds} \) is directed toward the center of curvature, i.e., it may point either toward \( \Omega_1 \) or away from \( \Omega_1 \) (i.e., toward \( \Omega_0 \)) depending on which direction the curve \( \Gamma \) bends. Since \( \nu \) has a fixed orientation, the curvature \( \zeta = \zeta(s) \) in formula (21) should be taken with the sign (see, e.g., [18, Part 1]):

\[
\zeta(s) = \begin{cases} 
\left| \frac{d\tau}{ds} \right|, & \text{if } \frac{d\tau}{ds} \cdot \nu > 0, \\
-\left| \frac{d\tau}{ds} \right|, & \text{if } \frac{d\tau}{ds} \cdot \nu < 0.
\end{cases}
\] (22)

To define the coordinates associated with the curve \( \Gamma \) we take into account that the shortest path from a given node that belongs to the grid boundary \( \gamma_1 \) or \( \gamma_0 \) to the curve \( \Gamma \) is along the normal. Denote the value of the parameter of the curve at the foot of this normal as \( s \), and the distance between the original point and the foot of the normal as \( n \), see Figure 3. As the position of the point may be on either side of the curve,
the value of the distance \( n \) is taken with the sign: \( n > 0 \) corresponds to the positive direction \( \mathbf{\nu} \), i.e., to the exterior of \( \Omega_1 \) (toward \( \Omega_0 \)), and \( n < 0 \) corresponds to the negative direction of \( \mathbf{\nu} \), i.e., to the interior of \( \Omega_1 \). The pair of numbers \((n, s)\) provides the orthogonal coordinates that identify the location of a given point on the plane.

For a general shape of the boundary \( \Gamma \) the coordinates \((n, s)\) may be prone to some ambiguity, as multiple shortest normals may exist for some of the nodes. Therefore, the multi-valued distance function at such nodes is non-differentiable with respect to the arc length \( s \). The multiple shortest distances for a given node may occur when the minimum radius of curvature \( \bar{R} = \min_s R(s) \) is of order \( h \) since the coordinates \((n, s)\) are used only for the points of the grid boundary \( \gamma \) which are all about one grid size \( h \) away from the curve \( \Gamma \), see Figures 1, 2 and 3. This implies that the grid does not adequately resolve the geometry, and needs to be refined. The simulations in this paper, see Section 5, do not involve shapes with the features where the curvature \( \zeta \sim h^{-1} \). In the future, we hope to analyze shapes with “small” features.

The formulae hereafter are all extensions of those obtained in [13, 14] for circular and elliptical obstacles. The coordinates \((n, s)\) are orthogonal but not necessarily orthonormal. For a given point \((n, s)\), its radius-vector \( r \) is expressed as follows:

\[
\mathbf{r} = \mathbf{r}(n, s) = R(s) + n\mathbf{\nu}(s) = \left( R_x(s) + n\nu_x(s), R_y(s) + n\nu_y(s) \right)
\]

\[
= \left( R_x + n\frac{dR_y}{ds}, R_y - n\frac{dR_x}{ds} \right).
\]

Consequently, the basis vectors are given by

\[
\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial n} = \left( \frac{dR_y}{ds}, -\frac{dR_x}{ds} \right) = (\tau_y, -\tau_x) = \mathbf{\nu}
\]

and

\[
\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial s} = \left( \frac{dR_x}{ds} + n\frac{d^2R_y}{ds^2}, \frac{dR_y}{ds} - n\frac{d^2R_x}{ds^2} \right)
\]

\[
= \left( \tau_x - n\zeta\tau_y, \tau_y - n\zeta\tau_x \right) = (1 - n\zeta)\tau,
\]

where we have used formulae (20), (21), and (22). Accordingly, the Lame coefficients for the coordinates \((n, s)\) are

\[
H_1 \equiv H_n = |\mathbf{e}_1| = 1
\]

and

\[
H_2 \equiv H_s = |\mathbf{e}_2| = |1 - n\zeta| = 1 - n\zeta,
\]  \hspace{1cm} (23)

where the last equality in (23) holds because \( n < \zeta^{-1} \) for \( \zeta > 0 \) and \( n > \zeta^{-1} \) for \( \zeta < 0 \) (otherwise, the minimum radius of curvature \( \bar{R} \) may be of order \( h \) or smaller).

In the coordinates \((n, s)\), the Helmholtz equation becomes

\[
\frac{1}{H_s} \left[ \frac{\partial}{\partial n} \left( H_s \frac{\partial u}{\partial n} \right) + \frac{\partial}{\partial s} \left( \frac{1}{H_s} \frac{\partial u}{\partial s} \right) \right] + k^2(n, s)u = f,
\]  \hspace{1cm} (24)
where $H_s = H_s(n, s)$ is given by (23), and we have taken into account that $H_n \equiv 1$. Equation (24) will be used for building the equation-based extension of a given $\xi_\Gamma$, from the continuous boundary $\Gamma$ to the nodes of the grid boundaries $\gamma_1$ and $\gamma_0$. If, in particular, $\Gamma$ is a circle of radius $R$, then the foregoing general constructs transform into the corresponding constructs for polar coordinates [13, 14]. In this case, the curvature $\zeta$ of (22) does not depend on $s$:

$$\zeta = -\frac{1}{R},$$

and consequently [see formula (23)],

$$H_s = 1 + \frac{n}{R} = \frac{R + n}{R} = \frac{r}{R}.$$

Then, according to (24), we can write:

$$\Delta u = \frac{R}{r} \left[ \frac{\partial}{\partial n} \left( \frac{r}{R} \frac{\partial u}{\partial n} \right) + \frac{\partial}{\partial s} \left( \frac{R}{r} \frac{\partial u}{\partial s} \right) \right] = \frac{1}{r} \frac{\partial}{\partial n} \left( \frac{r}{R} \frac{\partial u}{\partial n} \right) + \frac{R^2 \partial^2 u}{r^2 \partial s^2}.$$

Finally, we have $n = r - R$ so that $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$, and $s = R\theta$ so that $\frac{\partial}{\partial s} = \frac{1}{R} \frac{\partial}{\partial \theta}$, which yields:

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R} \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

### 4.2 Equation-based extension

Given $\xi_\Gamma$, we define a new smooth function $v = v(n, s)$ in the vicinity of $\Gamma$ by means of the Taylor formula:

$$v(n, s) = v(0, s) + \sum_{l=1}^{L} \frac{1}{l!} \frac{\partial^l v(0, s)}{\partial n^l} n^l. \quad (25)$$

The zeroth and first order derivatives in (25) coincide with the respective components of $\xi_\Gamma$:

$$v(0, s) = \xi_0(s) \quad \text{and} \quad \frac{\partial v(0, s)}{\partial n} = \xi_1(s).$$

All higher order derivatives in formula (25) are determined with the help of equation (24) applied to $v$. We multiply both sides of (24) by $H_s$ and obtain

$$H_s \frac{\partial^2 v}{\partial n^2} + \frac{\partial H_s}{\partial n} \frac{\partial v}{\partial n} + \left( \frac{\partial}{\partial s} \frac{1}{H_s} \right) \frac{\partial v}{\partial s} + \frac{1}{H_s} \frac{\partial^2 v}{\partial s^2} + H_s k^2 v = H_s f,$$

where $\frac{\partial H_s}{\partial n} = -\zeta(s)$ and $\frac{\partial}{\partial n} H_s = \frac{n}{H_s^2} \zeta'(s)$, see formula (23). Then, we solve for the second derivative with respect to $n$, which yields:

$$\frac{\partial^2 v}{\partial n^2} = f(n, s) - k^2(n, s) v + \frac{\zeta}{H_s} \frac{\partial v}{\partial n} - \frac{n \zeta'(s)}{H_s^3} \frac{\partial v}{\partial s} - \frac{1}{H_s} \frac{\partial^2 v}{\partial s^2}. \quad (26)$$

Consequently,

$$\frac{\partial^2 v(0, s)}{\partial n^2} = f(0, s) - k^2(0, s) \xi_0(s) + \zeta(s) \xi_1(s) - \frac{\partial^2 \xi_0(s)}{\partial s^2}.$$
Next, we differentiate equation (26) with respect to $n$:

\[
\frac{\partial^3 v}{\partial n^3} = \frac{\partial f}{\partial n} - k^2 \frac{\partial v}{\partial n} - 2k \frac{\partial k}{\partial n} v + \left( \frac{\partial}{\partial n} \frac{1}{H_s} \right) \zeta \frac{\partial v}{\partial n} + \zeta \frac{\partial^2 v}{\partial n^2}
\]

\[
- \zeta' \left( \frac{1}{H_s^3} + n \left( \frac{\partial}{\partial n} \frac{1}{H_s^3} \right) \right) \frac{\partial v}{\partial s} - \left( \frac{\partial}{\partial n} \frac{1}{H_s^2} \right) \frac{\partial^2 v}{\partial s^2} - \frac{n \zeta'}{H_s^3} \frac{\partial^2 v}{\partial n \partial s} - \frac{1}{H_s^2} \frac{\partial^3 v}{\partial n \partial s^2}
\]

\[
= \frac{\partial f}{\partial n} - k^2 \frac{\partial v}{\partial n} - 2k \frac{\partial k}{\partial n} v + \zeta^2 \frac{\partial v}{\partial n} + \zeta \frac{\partial^2 v}{\partial n^2}
\]

\[
- \zeta' \left( \frac{1}{H_s^3} + n \left( \frac{\partial}{\partial n} \frac{1}{H_s^3} \right) \right) \frac{\partial v}{\partial s} - \frac{2 \zeta' \frac{\partial^2 v}{\partial s^2}}{H_s^3} - \frac{n \zeta'}{H_s^3} \frac{\partial^2 v}{\partial n \partial s} - \frac{1}{H_s^2} \frac{\partial^3 v}{\partial n \partial s^2}
\]

(27)

and substitute $n = 0$ to get:

\[
\frac{\partial^3 v(0, s)}{\partial n^3} = \frac{\partial f}{\partial n} - 2k \frac{\partial k}{\partial n} \xi_0(s) + (\zeta^2 - k^2) \xi_1(s) + \zeta \frac{\partial^2 v(0, s)}{\partial n^2} - \zeta' \frac{\partial^2 \xi_0(s)}{\partial s} - 2\zeta \frac{\partial^2 \xi_1(s)}{\partial s^2}.
\]

Similarly, the fourth normal derivative is obtained by differentiating (27) with respect to $n$:

\[
\frac{\partial^4 v}{\partial n^4} = \frac{\partial^2 f}{\partial n^2} - 2 \left( \left( \frac{\partial k}{\partial n} \right)^2 + k^2 \frac{\partial^2 k}{\partial n^2} \right) v + \left( \frac{2 \zeta^3}{H_s^3} - 4k \frac{\partial k}{\partial n} \right) \frac{\partial v}{\partial n} + \left( \frac{2 \zeta^2}{H_s^3} - k^2 \right) \frac{\partial^2 v}{\partial n^2}
\]

\[
+ \zeta \frac{\partial^3 v}{\partial n^3} - 3\zeta' \frac{\partial^3 v}{\partial n^3} \frac{\partial v}{\partial s} + n \left( \frac{\partial^2 v}{\partial s \partial n} + 3\zeta \frac{\partial v}{\partial s} \right) - \frac{6\zeta^2}{H_s^4} \frac{\partial^2 v}{\partial s^2}
\]

\[
- \frac{\zeta'}{H_s^3} \left( 2 \frac{\partial^2 v}{\partial s^2} + 3\zeta \frac{\partial v}{\partial s} + n \left( \frac{\partial^3 v}{\partial s^2 \partial n^2} + 3\zeta \frac{\partial^2 v}{\partial s^2} + 3\zeta' \frac{\partial^2 v}{\partial s^2} \right) \right)
\]

\[
- \frac{4\zeta}{H_s^3} \frac{\partial^3 v}{\partial n \partial s^2} - \frac{1}{H_s^2} \frac{\partial^4 v}{\partial n^2 \partial s^2}
\]

and substituting $n = 0$, which yields:

\[
\frac{\partial^4 v(0, s)}{\partial n^4} = \frac{\partial^2 f}{\partial n^2} - 2 \left( \left( \frac{\partial k}{\partial n} \right)^2 + k^2 \frac{\partial^2 k}{\partial n^2} \right) \xi_0(s) + \left( 2\zeta^3 - 4k \frac{\partial k}{\partial n} \right) \xi_1(s) + \left( 2\zeta^2 - k^2 \right) \frac{\partial^2 v(0, s)}{\partial n^2}
\]

\[
+ \zeta \frac{\partial^3 v(0, s)}{\partial n^3} - 6\zeta \frac{\partial^2 \xi_0(s)}{\partial s} - 6\zeta^2 \frac{\partial^2 \xi_0(s)}{\partial s^2} - 2\zeta \frac{\partial^2 \xi_1(s)}{\partial s} - 4\zeta \frac{\partial^2 \xi_1(s)}{\partial s^2} - \frac{\partial^4 v(0, s)}{\partial n^2 \partial s^2}.
\]

The quantity $\frac{\partial^4 v(0, s)}{\partial n^2 \partial s^2}$ is derived by differentiating (26) twice with respect to $s$:

\[
\frac{\partial^4 v}{\partial n^2 \partial s^2} = \frac{\partial^2 f}{\partial s^2} - 2 \left( \left( \frac{\partial k}{\partial s} \right)^2 + k^2 \frac{\partial^2 k}{\partial s^2} \right) v + \left( \frac{\zeta''}{H_s^2} + \frac{2n (\zeta')^2}{H_s^3} \right) (H_s + n\zeta') \frac{\partial v}{\partial s}
\]

\[
- \frac{n}{H_s^4} \left( H_s \zeta''' + 9n\zeta' \zeta'' + \frac{12n^2}{H_s^4} (\zeta')^3 \right) + 4k \frac{\partial k}{\partial s} \frac{\partial v}{\partial s} - \left( \frac{4n\zeta''}{H_s^3} + \frac{12n^2 (\zeta')^2}{H_s^4} + k^2 \right) \frac{\partial^2 v}{\partial s^2}
\]

\[
- \frac{5n\zeta'' \frac{\partial^3 v}{\partial s^3}}{H_s^3 \partial s} + \frac{1}{H_s^2 \partial s^4} + 2 \left( \frac{\zeta'}{H_s} + \frac{n\zeta''}{H_s^2} \right) \frac{\partial^2 v}{\partial n \partial s} + \zeta \frac{\partial^3 v}{\partial n \partial s^2}
\]
and substituting $n = 0$:

$$\frac{\partial^4 v(0, s)}{\partial n^2 \partial s^2} = \frac{\partial^2 f}{\partial s^2} - 2 \left( \left( \frac{\partial k}{\partial s} \right)^2 + k \frac{\partial^2 k}{\partial s^2} \right) \xi_0(s) + \zeta'' \frac{\partial v}{\partial n}$$

$$- 4k \frac{\partial k}{\partial s} \frac{\partial \xi_0(s)}{\partial s} - k^2 \frac{\partial^2 \xi_0(s)}{\partial s^2} - \frac{\partial^4 \xi_0(s)}{\partial s^4} + 2\zeta' \frac{\partial \xi_1(s)}{\partial s} + \zeta \frac{\partial^2 \xi_1(s)}{\partial s^2}.$$  

Higher order derivatives (e.g., for the sixth order scheme) can be obtained in the same manner.

We emphasize that formula (25) is not an approximation of a given $v(n, s)$ by its truncated Taylor’s expansion. It is rather the definition of a new function $v(n, s)$. This function is used for building the equation-based extension of $\xi_\Gamma$ from $\Gamma$ to $\gamma_j$, $j = 0, 1$:

$$\xi_{\gamma_j} = E x^{(j)} \xi_\Gamma \overset{\text{def}}{=} v(n, s)|_{\gamma_j}. \quad (28)$$

In other words, extension (28) is obtained by drawing a normal from a given node of $\gamma_j$ to $\Gamma$, see Figure 3, and then using the Taylor formula with higher order derivatives computed by differentiating the governing equation (24).

## 5 Results

We consider the following problem [cf. formula (1)]:

$$\begin{cases}
\Delta u + k_0^2 u = 0 & x \in \Omega_0, \\
\Delta u + k_1(x)^2 u = 0 & x \in \Omega_1.
\end{cases} \quad (29)$$

Driven by the incident wave $u^{(\text{inc})} = e^{ik_0 R_x \cos \theta + ik_0 R_y \sin \theta}$, where $\theta$ denotes the angle of incidence. The wavenumber for the exterior domain $\Omega_0$ is constant while the wavenumber for the interior domain $\Omega_1$ varies:

$$k_1(x) = \tilde{k}_1 e^{-10(r-r_0)^6}, \quad (30)$$

where $r_0 = 1.6$ and $\tilde{k}_1$ is a parameter that assumes different values for different simulations described below. In what follows, we solve problem (29) for several irregular shapes of the interface $\Gamma = \{ R(s) = (R_x(s), R_y(s)) \}$ between $\Omega_0$ and $\Omega_1$. Since the problems we solve has no known exact solution, the error is measured for between the current solution and the solution on the subsequent grid, i.e.

$$||u^h - u^{2h}||_\infty.$$  

### 5.1 A star with rounded edges

The first case is an interface shaped as a five-point star with rounded edges to make it smooth. It is shown in Figure 4 and is given by the following parametric expression:

$$\Gamma = R(t) = (R_x(t), R_y(t)) = \left( \frac{1}{6} \cos(4t) + \frac{7}{6} \sin(t), \frac{7}{6} \cos(t) + \frac{1}{6} \sin(4t) \right), \quad 0 \leq t \leq 2\pi.
For the interior auxiliary problem, we choose a Cartesian grid on the square $[-1.7, 1.7] \times [-1, 7, 1.7]$. For the exterior AP, we choose a polar grid on the annulus $\{0.3 \leq r \leq 2.2\}$. The grid boundaries $\gamma_1$ and $\gamma_0$ defined by formulae (7) and (9), respectively, are also shown in Figure 4 for the case where the dimension of the main discretization grid is $33 \times 33$. Note, the auxiliary domain should allow for at least a few grid nodes between $\Gamma$ and the outer boundary, so that the grid boundary $\gamma_1$ or $\gamma_0$ is fully inside the grid. The variable wave number for the entire domain is shown in Figure 5: specifically, in Figure 5(a) the wave numbers are $k_0 = 1$ and $k_1 = k_1(x)$ with $\tilde{k}_1 = 3$, see formula (30), and in Figure 5(b) we have $k_0 = 5$ and $\tilde{k}_1 = 10$. In Figure 6, we present the solution for the incident plane wave at $\theta = 0^\circ$. Table 1 demonstrates the grid convergence for various sets of parameters.

![Figure 4](image-url)

(a) Cartesian grid  
(b) Polar grid

Figure 4: The grid boundaries for the star with rounded edges on a $33 \times 33$ grid.
Figure 5: Profile of the variable wave number $k$ for the star with rounded edges.

(a) $k_0 = 1$ and $\tilde{k}_1 = 3$

(b) $k_0 = 5$ and $\tilde{k}_1 = 10$
Figure 6: Total field for the transmission and scattering of a plane wave about a star with rounded edges at the angle of incidence $\theta = 0^\circ$, with $k_0 = 5$ and $\tilde{k}_1 = 10$. 

(a) Absolute value

(b) Imaginary part

(c) Real part
For the interior auxiliary problem, we take a Cartesian grid on the rectangle $[-7, 7] \times [-7, 7]$. For the exterior AP, we take a polar grid on the annulus $\{0.8 \leq r \leq 2.2\}$. The continuous boundary $\Gamma$ and the grid boundaries $\gamma_1$ and $\gamma_0$ are shown in Figure 7. The wave number for the kite is shown in Figure 8 for two different cases: $k_0 = 1$ and $k_1 = k_1(x)$ given by (30) with $k_1 = 3$ in Figure 8(a), and $k_0 = 5$, $k_1 = 10$ in Figure 8(b). In Figure 9 we display the solution driven by the plane wave at an angle of incidence $\theta = 0^\circ$. Table 2 presents the results for the grid convergence for three different choices of the exterior and interior wave number.

Table 1: Fourth order grid convergence for the transmission and scattering of a plane wave with the incidence angle $\theta = 0^\circ$ about a star with rounded edges.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Interior</th>
<th>Exterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0 = 1, \tilde{k}_1 = 3, M = 89$</td>
<td>$</td>
<td>u^h - u^{2h}</td>
</tr>
<tr>
<td>$64 \times 64$</td>
<td>1.897713e + 01 ---</td>
<td>1.359849e + 01 ---</td>
</tr>
<tr>
<td>$128 \times 128$</td>
<td>6.274128e + 00 1.60</td>
<td>6.472424e + 00 1.07</td>
</tr>
<tr>
<td>$256 \times 256$</td>
<td>2.693255e - 03 11.19</td>
<td>7.835759e - 03 9.69</td>
</tr>
<tr>
<td>$512 \times 512$</td>
<td>1.266509e - 05 7.73</td>
<td>1.072922e - 04 6.19</td>
</tr>
<tr>
<td>$1024 \times 1024$</td>
<td>6.169030e - 07 4.36</td>
<td>4.556798e - 06 4.56</td>
</tr>
<tr>
<td>$2049 \times 2049$</td>
<td>3.215375e - 08 4.26</td>
<td>3.142369e - 07 3.86</td>
</tr>
</tbody>
</table>

5.2 A kite

Next, we consider an interface in the form of a kite given by:

$$\Gamma = R(t) = (R_x(t), R_y(t)) = (\cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t), \quad 0 \leq t \leq 2\pi.$$
Figure 7: The grid boundaries for the kite on a $33 \times 33$ discretization grid.

Table 2: Fourth order grid convergence for the transmission and scattering of a plane wave with the incidence angle $\theta = 0^\circ$ about a kite.
Figure 8: The wave number $k$ for the kite.
Figure 9: Total field for the transmission and scattering of a plane wave about a kite at the angle of incidence $\theta = 0^\circ$, with $k_0 = 5$ and $k_1 = 10$. 
5.3 A submarine-like scatterer

The third case is a submarine-like interface defined by:

\[ \Gamma = R(t) = (R_x(t), R_y(t)) = \left( 1.8 \cos t, 0.36 \sin t \cdot \left( 1 + 2 \cdot \left( \frac{\cos \frac{t}{2} + \sin \frac{t}{2}}{\sqrt{2}} \right)^{150} \right) \right), \]

where \( 0 \leq t \leq 2\pi \). The interior AP is solved on the rectangle \([-2.2, 2.2] \times [-0, 6, 1.2]\) using a Cartesian grid. The exterior AP is solved on a polar grid in the annulus \( \{0.3 \leq r \leq 2.2\} \). Figure 10 presents the geometry of the interface \( \Gamma \) and the grid sets. Figure 11 shows the variation of the wave number \( k \) across the computational domain for two cases: \( k_0 = 1 \) and \( k_1 = k_1(x) \) with \( \bar{k}_1 = 3 \) (see formula (30)) is in Figure 11(a), and \( k_0 = 5, k_1 = k_1(x) \) with \( \bar{k}_1 = 10 \) is in Figure 11(b).

For the case of a submarine-like body, we solve two problems: an external scattering problem with a homogeneous Dirichlet boundary condition at the surface \( \Gamma \) and a transmission/scattering problem similar to that solved in Section 5.1 for the star interface and in Section 5.2 for the kite interface. The external scattering solution for \( k_0 = 10 \) and \( \theta = 0^\circ \) is presented in Figure 12. Table 3 demonstrates the grid convergence for three different choices of the exterior wave number. The transmission/scattering solution for the submarine is shown in Figure 13, while Table 4 summarizes the grid convergence results for this case.

![Cartesian grid and Polar grid](image)
Figure 11: The wave number $k$ for the submarine.
Figure 12: Scattered field for the external scattering of a plane wave about a submarine at the angle of incidence $\theta = 0^\circ$ and with $k_0 = 10$. 
Figure 13: Total field for the transmission and scattering of a plane wave about a submarine at the angle of incidence $\theta = 0^\circ$, with $k_0 = 5$ and $\tilde{k}_1 = 10$.

6 Conclusions

We have described a combined implementation of the method of difference potentials together with a compact high order accurate finite difference scheme for the numerical solution of wave
and polar grids. Such problems) for general shaped domains with a varying wavenumber using only Cartesian derivative directly at the interface (without having to interpolate and/or use one-sided differences, the proposed methodology are its capability to accurately reconstruct the solution and/or its normal equation is approximated on a regular structured grid, which is efficient and entails a low computational complexity. At the same time, the method of difference potentials guarantees no loss of accuracy for curvilinear non-conforming boundaries. We can also handle variable coefficients that describe a non-homogeneous medium. Thus, this methodology provides a viable alternative to computational propagation problems in the frequency domain for the case of general geometries. The Helmholtz equation is approximated on a regular structured grid, which is efficient and entails a low computational complexity. At the same time, the method of difference potentials guarantees no loss of accuracy for curvilinear non-conforming boundaries. We can also handle variable coefficients that describe a non-homogeneous medium. Thus, this methodology provides a viable alternative to both boundary element methods and high order finite element methods. Among the advantages of the proposed methodology are its capability to accurately reconstruct the solution and/or its normal derivative directly at the interface (without having to interpolate and/or use one-sided differences, such as done in conventional finite differences and finite elements).

The performance of our method and its design high order accuracy have been corroborated numerically by solving a variety of 2D transmission/scattering problems (as well as pure external scattering problems) for general shaped domains with a varying wavenumber using only Cartesian and polar grids.

<table>
<thead>
<tr>
<th>Grid</th>
<th>( k_0 = 1, M = 95 )</th>
<th>( k_0 = 5, M = 100 )</th>
<th>( k_0 = 10, M = 105 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(</td>
<td>u^h - u^{2h}</td>
<td>_\infty ) rate</td>
</tr>
<tr>
<td>64 ( \times ) 64</td>
<td>1.051374e + 08</td>
<td>1.052972e + 07</td>
<td>8.293744e + 05</td>
</tr>
<tr>
<td>128 ( \times ) 128</td>
<td>1.259820e + 01</td>
<td>22.99</td>
<td>6.467906e + 01</td>
</tr>
<tr>
<td>256 ( \times ) 256</td>
<td>6.211242e - 03</td>
<td>10.99</td>
<td>1.791608e - 02</td>
</tr>
<tr>
<td>512 ( \times ) 512</td>
<td>6.999940e - 04</td>
<td>3.16</td>
<td>1.789315e - 03</td>
</tr>
<tr>
<td>1024 ( \times ) 1024</td>
<td>4.196957e - 05</td>
<td>4.05</td>
<td>9.784597e - 05</td>
</tr>
<tr>
<td>2048 ( \times ) 2048</td>
<td>1.841662e - 06</td>
<td>4.51</td>
<td>5.761461e - 06</td>
</tr>
</tbody>
</table>

Table 3: Fourth order grid convergence for the external scattering of a plane wave about a submarine at the angle of incidence \( \theta = 0^\circ \).

| Grid | Exterior | \( k_0 = 1, \tilde{k}_1 = 3, M = 90 \) | \( k_0 = 5, \tilde{k}_1 = 10, M = 95 \) | \( k_0 = 10, \tilde{k}_1 = 20, M = 100 \) |
|------|----------------|----------------|----------------|
|      | \( |u^h - u^{2h}|_\infty \) rate | \( |u^h - u^{2h}|_\infty \) rate | \( |u^h - u^{2h}|_\infty \) rate |
| 64 \( \times \) 64 | 2.295752e + 01 | 2.237487e + 00 | 9.167262e - 01 |
| 128 \( \times \) 128 | 1.375159e + 02 | -2.58 | 1.619068e + 03 | -9.50 | 3.583345e + 05 | -18.46 |
| 256 \( \times \) 256 | 4.990559e - 01 | 8.11 | 3.93502e + 00 | 8.68 | 2.036714e + 00 | 17.42 |
| 512 \( \times \) 512 | 1.290265e - 03 | 8.60 | 8.432259e - 03 | 8.87 | 3.475999e - 02 | 5.87 |
| 1024 \( \times \) 1024 | 1.293954e - 05 | 6.64 | 6.619289e - 05 | 6.99 | 1.481051e - 04 | 7.87 |
| 2049 \( \times \) 2049 | 4.003078e - 07 | 5.01 | 2.210780e - 06 | 4.90 | 5.048151e - 06 | 4.87 |

| Grid | Interior | \( |u^h - u^{2h}|_\infty \) rate | \( |u^h - u^{2h}|_\infty \) rate | \( |u^h - u^{2h}|_\infty \) rate |
|------|----------------|----------------|----------------|
|      | \( |u^h - u^{2h}|_\infty \) rate | \( |u^h - u^{2h}|_\infty \) rate | \( |u^h - u^{2h}|_\infty \) rate |
| 64 \( \times \) 64 | 7.864884e + 00 | 9.351113e - 01 | 9.979765e - 01 |
| 128 \( \times \) 128 | 4.266917e + 03 | -9.08 | 2.537239e + 05 | -18.05 | 3.109921e + 07 | -24.89 |
| 256 \( \times \) 256 | 5.132027e - 01 | 13.02 | 5.635594e + 00 | 15.46 | 2.392006e + 00 | 23.63 |
| 512 \( \times \) 512 | 1.713991e - 03 | 8.23 | 2.500136e - 02 | 7.82 | 7.836997e - 02 | 4.93 |
| 1024 \( \times \) 1024 | 6.527788e - 05 | 4.71 | 7.829261e - 04 | 5.00 | 2.928307e - 03 | 4.74 |
| 2049 \( \times \) 2049 | 5.213601e - 07 | 6.97 | 2.894848e - 06 | 8.08 | 7.650070e - 06 | 8.58 |

Table 4: Fourth order grid convergence for the transmission and scattering of a plane wave about a submarine at the angle of incidence \( \theta = 0^\circ \).
References


