| 0 | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| /4 | /24 | /24 | /24 | /24 | /100 |

**0.**(4pt)Read the cover and write your name and ID number on the cover.(**PRINT**. If I can't read them, you may loose points.)

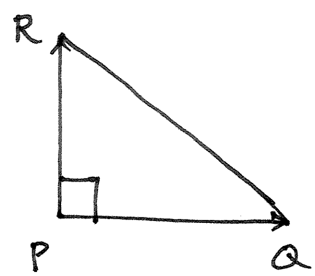
**1.**(12pt+12pt) The three points are given by $P(0,1,-1)$, $Q(1,0,-2)$, and $R(1,3,-2)$.

(a)Calculate the position vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$\overrightarrow{PQ} = \langle 1,0,-2\rangle - \langle 0,1,-1\rangle = \langle 1,-1,-1\rangle$

$\overrightarrow{PR} = \langle 1,3,-2\rangle - \langle 0,1,-1\rangle = \langle 1,2,-1\rangle$

(b)Show that the triangle $PQR$ is a right triangle.




$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 1 - 2 + 1 = 0$

So, the angle at $P$ is $\frac{\pi}{2}$.

**2.**(10pt+10pt+4pt) Two points are given by $P(3,5,4)$ and $Q(0,1,1)$.
(a)Calculate the line equation for the line $L$ which passes both $P$ and $Q$.

Let $\vec{v} = \overrightarrow{QP} = \langle 3,5,4\rangle - \langle 0,1,1\rangle = \langle 3,4,3\rangle$.
Then we can use $\vec{v}$ as the direction vector.
The line passes $Q(0,1,1)$. So, the parametric equations are

$$\begin{cases} x = 3t + 0 = 3t \\ y = 4t + 1 \\ z = 3t + 1 \end{cases}$$

(b)Suppose that $(1, y_1, z_1)$ is on the line L. Calculate $y_1$ and $z_1$.

If $x = 1$, $3t = 1$. Therefore, $t = \frac{1}{3}$.
the Corresponding $y$ component is $y = 4\cdot\frac{1}{3} + 1 = \frac{7}{3}$
and $z = 3\cdot\frac{1}{3} + 1 = 2$

$y_1 = \frac{7}{3}$, $z_1 = 2$

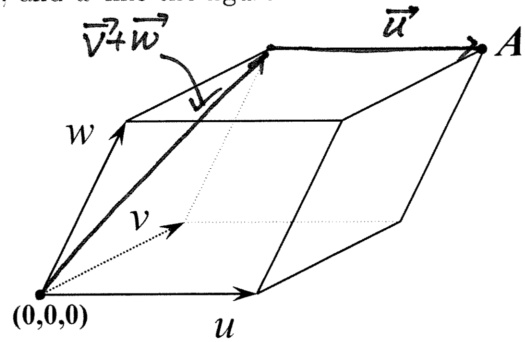
(c)Calculate the distance from $(1, y_1, z_1)$ to the plane $P_0 : x + y + 2z = 1$.

$$\text{distance} = \frac{|1 + y_1 + 2z_1 - 1|}{\sqrt{1+1+4}} = \frac{\left|\frac{7}{3} + 2\cdot 2\right|}{\sqrt{6}}$$

$$= \frac{19}{3\sqrt{6}}$$

**3.**(24pt)$\vec{u} =< 1, 2, 3 >, \vec{v} =< 2, 1, 3 >, \vec{w} =< 3, 2, 6 >$.

(a)(12pt)Consider the parallelopiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$ like the figure. Calculate the coordinates of the vertex $A$.




~~Key~~

$$\vec{OA} = (\vec{v}+\vec{w}) + \vec{u}$$

$$= \langle 6, 5, 12 \rangle$$

$(6, 5, 12)$ is the coordinate.

(b)(12pt)Calculate the volume of the parallelopiped.

$$\text{Volume} = |\vec{w} \cdot (\vec{u} \times \vec{v})|$$

$$= \left| \begin{vmatrix} 3 & 2 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{vmatrix} \right| = |3 \cdot 3 - 2(-3) + 6 \cdot (-3)|$$

$$= |9 + 6 - 18|$$

$$= 3$$

$$\vec{r}' = \left\langle -\tfrac{3}{5}\sin t,\ -\cos t,\ \tfrac{4}{5}\sin t \right\rangle$$

$$|r'| = \sqrt{\tfrac{9}{25}\sin^2 t + \cos^2 t + \tfrac{16}{25}\sin^2 t} = 1$$

$$\vec{r}'' = \left\langle -\tfrac{3}{5}\cos t,\ \sin t,\ \tfrac{4}{5}\cos t \right\rangle$$

**4.**(10pt+10pt+4pt) A curve is given by

$$\vec{\mathbf{r}}(t) = < \frac{3}{5}\cos t, 1 - \sin t, -\frac{4}{5}\cos t > .$$

(a)Calculate $\mathbf{T}$ and $\kappa$.($\mathbf{T}$ is the unit tangent vector and $\kappa$ is the curvature.)

$$\vec{T} = \frac{r'}{|r'|} = r' = \left\langle -\tfrac{3}{5}\sin t, -\cos t, \tfrac{4}{5}\sin t \right\rangle$$

$$\kappa = \frac{|r' \times r''|}{1^3} = |r' \times r''| = \cancel{\tfrac{12}{25}\sin t\cos t + \tfrac{12}{25}\sin t\cos t}$$

$$= \left| \left\langle -\tfrac{4}{5}\cos^2 t - \tfrac{4}{5}\sin^2 t,\ -\tfrac{12}{25}\sin t\cos t + \tfrac{12}{25}\sin t\cos t,\ -\tfrac{3}{5}\sin^2 t - \tfrac{3}{5}\cos^2 t \right\rangle \right|$$

$$= \left| \left\langle -\tfrac{4}{5}, 0, -\tfrac{3}{5} \right\rangle \right| = 1$$

(b)Calculate $a_T$, $a_N$, and $\mathbf{N}$.($\mathbf{N}$ is the unit normal vector, and $a_T$ and $a_N$ are the coefficients in $\vec{\mathbf{r}}''(t) = a_T\mathbf{T} + a_N\mathbf{N}$.)

$$a_T = \frac{r' \cdot r''}{|r'|} = \frac{\tfrac{9}{25}\sin t\cos t - \sin t\cos t + \tfrac{16}{25}\sin t\cos t}{1} = 0$$

$$a_N = \frac{|r' \times r''|}{|r'|} = |r' \times r''| = 1 \quad (\text{same as } \kappa)$$

$$N = \frac{r'' - a_T T}{a_N} = \frac{r'' - 0T}{1} = r'' \quad (\text{see above})$$

(c)Show that this curve $\vec{\mathbf{r}}$ is on a sphere.

$$|\vec{r} - \langle 0,1,0 \rangle| = \left| \left\langle \tfrac{3}{5}\cos t, -\sin t, \tfrac{4}{5}\cos t \right\rangle \right| = 1$$

So $\vec{r}$ is on a sphere with the center $\langle 0,1,0 \rangle$ and the radius 1