Optimal pentamodes for guiding stress

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Two Problems:

(1) Concentrating a field into a region.

(2) Shielding a region from fields.
Sharp corners concentrate fields
How to measure this?
Threshold exponents on $L^\gamma$ integrability:

$$\gamma^- \equiv \inf_\gamma \int_B |\mathbf{E}(x)|^\gamma \, dx < \infty$$

$$\gamma^+ \equiv \sup_\gamma \int_B |\mathbf{E}(x)|^\gamma \, dx < \infty$$

$B$ is any Ball containing $\Omega$.

Equivalently, given a (possibly disconnected) subregion $Q \subset \Omega$ of small subvolume $|Q|$ one can maximize or minimize

$$\int_Q |\mathbf{E}(x)|^2 \, dx$$

and ask how this depends on $|Q|$ asymptotically as $|Q| \to 0$
Two isotropic conductors, conductivities $\sigma_1, \sigma_2$.
Uniform field at infinity

Some Candidates:
Best:

GWM (1986)

Tree for the material
Beauty Contest (GWM, 1986):

Fig. 9. Comparison of threshold exponents for the laminate of Fig. 8.

( ), eqs. (4.13) and (4.18); an array of diamond shaped
grains ( ), eq. (4.8); a checkerboard of the two components
( ), eq. (4.9); and Schulgasser’s symmetric material,
both in three dimensions ( ) and in two dimensions
( ), eqs. (4.11) and (4.12).

Proof of this microstructure independent Lower Bound on $\gamma^+$: Morrey (1938);
Boyarski (1957)
Proof of this microstructure Upper Bound on $\gamma^-$: Leonetti and Nesi (1997)
See Also Faraco (2003)
What about 3d?
For a uniform applied field the local field can vanish between the torii, even at finite conductivity ratios.
It’s constantly a surprise to find what properties a composite can exhibit.

One interesting example:

\[ F = qv \times B \]
Non-symmetric conductivity matrix with the antisymmetric part proportional to $\mathbf{B}$

In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

However there is a counterexample!
Geometry suggested by artist Dylon Whyte

A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)
Experimental Realization of Kern, Kadic, Wegener
Back to the shielding problem:

It seems more reasonable to require that there is no microstructure in the shielded region and that the microstructure is localized in a box.
Using Disks:

Concentration

Shielding
Field between two highly conducting disks close to touching

\[ a = \frac{1}{2} \sqrt{(1 - 1/c^2)}. \]

McPhedran, Poladian, GWM (1988)

\[ B_1 = \frac{- (c/2) (1 - 1/c)}{2s \ln(c) + 1 - 2s [\gamma + \psi(1+s)]}. \]

\[ a = \frac{1}{2} \sqrt{(1 - 1/c^2)}. \quad a_\infty = \frac{1}{2}(1 - 1/c). \]

\[ \psi: \text{Psi or Digamma function} \]

Rigorous Analysis: Lim and Yu (2015)

\[ \rho_- (a^2/x) = -\eta \rho_+(x) \]
\[ \eta = (\sigma - 1)/(\sigma + 1). \]

\[ \rho_+(1-x) = -\rho_-(x), \]
\[ \rho_- [a^2/(1-x)] = \eta \rho_-(x) \]
\[ \rho_-(x) = A [(a_\infty - x)/(1 - a_\infty - x)]^s \]
\[ s = \ln(\eta)/\ln[a_\infty/(1 - a_\infty)] \]
Could use the transformation based approach of Greenleaf, Lassas, and Uhlmann.

Advantages: Works for any external field and creates no disturbance.

Disadvantages: Requires extreme conductivities, and if one truncates the solution there is no reason to expect it is optimal.
Or Maybe?

Seems like we are just guessing. Is there a more systematic approach, at least in the case where we use just 2 conducting materials, and we are seeking shielding or concentration for just one applied field?
Possible (average heat current, \( q^0 \), average temperature gradient, \( e^0 \)) pairs in a two phase conducting composite (Raitum, 1978).

\[
\nabla \cdot q = 0, \quad q(x) = k(x) e(x), \quad e = -\nabla T
\]

\( q, e \) periodic, \( \langle q \rangle = q^0, \quad \langle e \rangle = e^0 \),

Follows from the Wiener bounds:

\[

k^- I \leq k^* \leq k^+ I
\]

\[
k^+ = f k_1 + (1 - f) k_2
\]

\[
k^- = (f/k_1 + (1 - f)/k_2)^{-1}
\]

Solution of the "weak G-closure" problem for conductivity
The heat lens problem: Gibiansky, Lurie and Cherkaev (1988)

Aim: Shield or concentrate flux in the blue dashed interval

How does one optimally distribute a poor and good conductor to do this?
Field Shield: (Black, good conductor)

\[ x_2 = 1 \]
\[ x_2 = a \]
\[ q_1 = 1 \]
\[ q_2 = 0 \]
\[ x_1 = 0 \]
\[ x_2 = -a \]
\[ x_2 = -1 \]
\[ q_1 = 1 \]
\[ x_1 = 2w \]
Field Concentrator:
What if $k_0 = 0$?

Given $\mathbf{q}^0$ the weak G-closure provides a linear constraint on $\mathbf{e}^0$:

$$\mathbf{q}^0 \cdot \mathbf{q}^0 / (f_1 k_1) \leq \mathbf{q}^0 \cdot \mathbf{e}^0$$

The endpoint of $\mathbf{e}^0$ must lie to the right of the plane.

It is attained for laminate geometries but also wire geometries where the effective tensor takes the form:

$$\mathbf{k}^* = f_1 k_1 \mathbf{a} \otimes \mathbf{a}, \quad \mathbf{a} \cdot \mathbf{a} = 1$$

Makes sense: wires are best for conducting current
Many Solutions to the shielding problem:

The weak G-closure is still needed if we:

(1) Want to minimize the thermal resistance.

(2) Not use too much of the highly conducting phase (may, e.g., be expensive or heavy).
To solve similar optimization problems for elasticity, can we find the “weak G-Closure” for 3d-elastcity?

At least in the case for 3d printed materials when one phase is void and the other elastically isotropic?

A difficult problem: need to characterize possible (average strain $\epsilon^0$, average stress $\sigma^0$) pairs,

Can assume $\sigma^0$ is diagonal and normalized: 2 parameters
Then $\epsilon^0$ has 6 parameters.

So the “weak G-Closure” is described by a set in an 8-dimensional space, 11 if one includes the volume fraction, and bulk and shear moduli of the initial elastic material.
Problem:

\(\sigma(x), \epsilon(x)\) periodic,

\[
\nabla \cdot \sigma = 0, \quad \sigma(x) = C(x)\epsilon(x), \quad \epsilon = \left[\nabla u + (\nabla u)^T\right]/2.
\]

\[
C(x) = C_1\chi(x) + C_2(1 - \chi(x)), \quad \sigma^0 = \langle \sigma \rangle, \quad \epsilon^0 = \langle \epsilon \rangle, \quad f = \langle \chi \rangle
\]

Given \(f\) what is the range of values the pairs \((\sigma^0, \epsilon^0)\) take in the limit \(C_2 \to 0\) as the microgeometry varies \(\chi(x)\) varies over all possible configurations?

Characterizing possible elasticity tensors \(C_*\) of porous media a much HARDER problem as these live in an 18-dimensional space of tensor invariants. If we add the volume fraction and material moduli its a problem in a 21-dimensional space
One constraint implied by sharp bounds on the minimum compliance energy:

\[ W_f(\sigma^0) \leq \sigma^0 : \varepsilon^0, \quad (*) \]

Explicit expression for \( W_f(\sigma_0) \) given by Gibiansky and Cherkaev (1987) and Allaire (1994). Note \( W_f(cA) = c^2 W_f(A) \)

Our main result is that these optimal bounds on the compliance energy also provide optimal bounds on \((\varepsilon^0, \sigma^0)\)-pairs. Given \( \sigma^0 \) they constrain \( \varepsilon^0 \) to lie on one-side of a hyperplane.
Explicit Formula for Bound: (can skip)

$$W_f(\sigma^0) = \sigma^0 : C_1^{-1} \sigma^0 + \frac{f}{2\mu} g(C, \sigma^0), \quad \text{(Using Allaire’s notation.)}$$

Suppose the stress has eigenvalues $\sigma_1$, $\sigma_2$ and $\sigma_3$. Can assume at most one eigenvalue is negative, and $\sigma_1 \leq \sigma_2 \leq \sigma_3$. When all are non-negative, and $\lambda > 0$:

$$g(C, \sigma) = \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \leq \sigma_1 + \sigma_2,$$

$$= (\sigma_1 + \sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \geq \sigma_1 + \sigma_2,$$

while when one eigenvalue, namely $\sigma_1$, is negative,

$$g(C, \sigma) = \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} \left( \sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \sigma_1 \right)^2$$

$$\text{if } \sigma_3 + \sigma_2 \geq \frac{-\mu}{\mu + \lambda} \sigma_1 \text{ and } \sigma_3 - \sigma_2 \leq \frac{-\mu}{\mu + \lambda} \sigma_1,$$

$$= (\sigma_3 + \sigma_2)^2 + \sigma_1^2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 + \sigma_2 \leq \frac{-\mu}{\mu + \lambda} \sigma_1,$$

$$= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{2\mu}{\mu + \lambda} \sigma_1 \sigma_2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 - \sigma_2 \geq \frac{-\mu}{\mu + \lambda} \sigma_1.$$
The bound: very similar to the conductivity case when \( k_2 = 0 \).

Bounding surface

\[ W_f(\sigma^0) = \sigma^0 : \epsilon^0 \]

Response of the hierarchical laminate that has minimum compliance energy

6-dimensional space of symmetric 3 \( \times \) 3 matrices

Response surface of the "optimal" pentamode.

Response surface of an "ideal" pentamode.
The required geometries are pentmodes, materials with elastic tensor 

\[ \mathbf{C}^* = \alpha \mathbf{A} \otimes \mathbf{A}, \quad \mathbf{A} : \mathbf{A} = 1 \]

that are optimal in the sense that

\[ \alpha = 1/W_f(\mathbf{A}) \]

Given any \( \mathbf{\sigma}_0 \) and \( \mathbf{\epsilon}_0 \) so that (\*) holds as an equality, we choose

\[ \mathbf{A} = \mathbf{\sigma}_0 / \sqrt{\mathbf{\sigma}_0 : \mathbf{\sigma}_0} \]

and then

\[ \mathbf{C}^* \mathbf{\epsilon}_0 = \alpha \mathbf{\sigma}_0 W_f(\mathbf{\sigma}_0) / (\mathbf{\sigma}_0 : \mathbf{\sigma}_0) = \alpha \mathbf{\sigma}_0 W_f(\mathbf{A}) = \mathbf{\sigma}_0 \]

as desired.
What are pentamodes?

New classes of elastic materials (with Cherkaev, 1995)

A three dimensional pentamode material which can support any prescribed loading

Like a fluid it only supports one loading, unlike a fluid that loading may be anisotropic
Pentamode structures are a sort of anisotropic inhomogeneous fluid

\[ C(x) = A(x) \otimes A(x), \quad \nabla \cdot A = 0, \]

\[ \sigma(x) = C(x) \varepsilon(x), \quad \nabla \cdot \sigma = 0, \quad \varepsilon = \left[ \nabla u + (\nabla u)^T \right]/2 \]

have the solution

\[ \sigma(x) = \alpha A(x) \]

where \( \alpha = "a constant" \) is the analog of pressure, and

\[ \alpha = \text{Tr}[A(x) \nabla u], \]

constrains \( \nabla u \). Thus \( A(x) \) is a sort of anisotropic "compressibility"
Realization of Kadic et al. 2012
Cloak making an object “unfeelable”: Buckmann et. al. (2014)
Disadvantage: not only does the shear modulus go to zero as they are made more ideal, but also the bulk modulus goes to zero.
Modifying the pentamodes:
Idea of proof: Insert into the material attaining the energy bounds a thin walled structure with sets of parallel walls:

Inside the walls put the appropriate modified pentamode material. Thus we obtain an optimal pentamode attaining the energy bounds.
For elastically isotropic materials one has the Hashin-Shtrikman Bounds

\[
\kappa_* \geq f_1\kappa_1 + f_2\kappa_2 - \frac{f_1 f_2 (\kappa_1 - \kappa_2)^2}{f_2\kappa_1 + f_1\kappa_2 + 4\mu_2/3},
\]

\[
\mu_* \geq f_1\mu_1 + f_2\mu_2 - \frac{f_1 f_2 (\mu_1 - \mu_2)^2}{f_2\mu_1 + f_1\mu_2 + \mu_2(9\kappa_2 + 8\mu_2)/[6(\kappa_2 + 2\mu_2)]},
\]

\[
\kappa_* \leq f_1\kappa_1 + f_2\kappa_2 - \frac{f_1 f_2 (\kappa_1 - \kappa_2)^2}{f_2\kappa_1 + f_1\kappa_2 + 4\mu_1/3},
\]

\[
\mu_* \leq f_1\mu_1 + f_2\mu_2 - \frac{f_1 f_2 (\mu_1 - \mu_2)^2}{f_2\mu_1 + f_1\mu_2 + \mu_1(9\kappa_1 + 8\mu_1)/[6(\kappa_1 + 2\mu_1)]}.
\]

The optimal pentamode supporting hydrostatic stress \(\sigma^0 = I\), is a material that for fixed \(f_1 = 1 - f_2\) in the limit \(\kappa_2, \mu_2 \to 0\) attains the bulk modulus upper bound, yet has zero shear modulus, \(\mu_* = 0\).
Hashin-Shtrikman bounding box when one phase is void, and the volume fraction is prescribed

See also Ostanin, Ovchinnikov, Tozoni, and Zorin (results in 2d)
https://doi.org/10.1016/j.jmps.2018.05.018
We can go much further and go a long way to completely characterizing the G-closure of 3d (and 2d) printed materials.

Joint work with Marc Briane and Davit Harutyunyan
Thank You!

References:


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