The theory of field patterns

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Part 1: Main Idea, Joint work with Ornella Mattei

Part 2: Field Patterns in Temporal Laminates,
Joint work with Alexander Movchan, Natasha Movchan, and Hoai-Minh Nguyen
Outline

1. Space-time microstructures
2. Field patterns
3. Results
4. Part 2: Field patterns in temporal laminates
5. Future work
This talk is about a new mathematical object- a new sort of wave
Space-time microstructures

\[(a u_t)_t - (b u_x)_x = 0\]

**Static materials:** \(a = a(x)\) and \(b = b(x)\)

**Space-time microstructures:** \(a = a(x, t)\) and \(b = b(x, t)\)

**Activated materials:**

The property pattern moves

**Kinetic materials:**

The material itself moves

Realization of space-time microstructures

- Liquid crystals
- Ferroelectric, ferromagnetic materials
- Pump wave + small amplitudes waves: parametric resonance [e.g. Louisell & Quate (1958)]
- Transmission line with modulated inductance [e.g. Cullen (1958)]
- Experiments and more references in [Honey & Jones (1958)]
  ...
- Walking droplets [e.g. Couder et al. (2005), Couder & Fort (2006), Bush (2015)]
- Breaking reciprocity, artificial magnetism for photons [e.g. Fang et al. (2012), Boada et al. (2012), Celi et al. (2014), Yuan et al. (2016)]
- Time reversal [e.g. Fink (2016), Goussev et al. (2016)]
An example: space-time laminates

- Screening from long wave disturbances [Lurie (1997)]
- Energy conservation for low frequency waves [Lurie & Weekes (2003)]
- Energy exponential growth for high frequency waves [Cassedy (1967)]
- Homogenization for low frequencies [Lurie (1997)]
Another example: space-time checkerboards

- Limit cycles + energy exponential growth [Lurie & Weekes (2006), Lurie et al. (2009)]
- Linear shocks $\rightarrow$ Quantum mechanics??
- No homogenization in the classic sense!

[Lurie (2007)]
Field patterns arise in wave equations with a space-time microstructure, when the microstructure has the interesting feature that a disturbance propagating along a characteristic line, and subsequently interacting with the microstructure, does not evolve into a cascade of disturbances, but rather concentrates on a pattern of characteristic lines. This pattern is the field pattern!
Statement of an equivalent "conductivity" problem

2D Conductivity problem

\[ j(x) = \sigma(x) e(x), \quad \text{where} \quad \nabla \cdot j = 0, \quad e = -\nabla V, \]

\[ \sigma(x) = \chi(x) \sigma_1 + [1 - \chi(x)] \sigma_2 \]

\[ \sigma_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\beta_1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \alpha_2 & 0 \\ 0 & -\beta_2 \end{pmatrix}, \]

N.B. For the analogous dielectric problem–Hyperbolic materials!! [e.g. Fisher & Gould (1969), Naik et al. (2013), Korzeb et al. (2015)]

\[ \alpha_i \frac{\partial^2 V_i}{\partial x_1^2} = \beta_i \frac{\partial^2 V_i}{\partial x_2^2} \]

\[ x_1 \rightarrow x, \quad x_2 \rightarrow t \]

\[ V_i(x, t) = V_i^+(x - c_i t) + V_i^-(x + c_i t) \]

\[ c_i = \sqrt{\frac{\alpha_i}{\beta_i}} \]
Another way of thinking about the d’Alembert solution

Conducting wires
Transmission and initial conditions

- Transmission conditions at a space-time interface with slope $w$
  N.B. To have uniqueness and existence of the solution: [Lurie (1997)]
  \[(w^2 - c_1^2)(w^2 - c_2^2) \geq 0\]

  \[
  T.C. \left\{ \begin{array}{l}
  V_1 = V_2 \\
  n \cdot \sigma_1 \nabla V_1 = n \cdot \sigma_2 \nabla V_2
  \end{array} \right.
  \]

- Initial conditions
  \[
  I.C. \left\{ \begin{array}{l}
  V(x, 0) = H(x - a) \\
  j_2(x, 0) = \delta(x - a)j_0
  \end{array} \right.
  \]
Green function for a generic space-time microstructure
Green function for a special microstructure
Green function for another special microstructure
Geometry: Relation to Characteristic Lines

\[ c_2 \quad c_1 + c_2 \]

\[ t_0 \quad x_0 \quad \frac{c_2}{c_1} \]

(a) (b)

\[ \frac{c_1 + c_2}{2c_1} \]
Multidimensional nature of field patterns

\[ V(x, t) = \sum_{i=1}^{m} V_{\alpha_i}(x, t) \]

Multidimensional space: \( V(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} V_{\alpha_i}(x_i, t) \)

Multidimensional potential: \( \mathbf{V}(x, t) \)
The unit cell of the microstructure with aligned inclusions
The unit cell problem

\[ V_i^+(x, t) = a_i^+[1 - H(x - c_i t)], \quad V_i^-(x, t) = a_i^- H(x + c_i t) \]

\[ \mathbf{j}_i^+ = a_i^+ \sqrt{\alpha_i \beta_i} \begin{pmatrix} c_i \\ 1 \end{pmatrix} \delta(x - c_i t) \equiv a_i^+ \gamma_i \frac{1}{\sqrt{1 + c_i^2}} \begin{pmatrix} c_i \\ 1 \end{pmatrix} \delta(x - c_i t) \]

\[ \mathbf{j}_i^- = a_i^- \sqrt{\alpha_i \beta_i} \begin{pmatrix} -c_i \\ 1 \end{pmatrix} \delta(x + c_i t) \equiv a_i^- \gamma_i \frac{1}{\sqrt{1 + c_i^2}} \begin{pmatrix} -c_i \\ 1 \end{pmatrix} \delta(x - c_i t) \]

with \( \gamma_i = \sqrt{\alpha_i (\alpha_i + \beta_i)} \)
Symmetric dynamics
Antisymmetric dynamics

\[ \dot{j}_0 \]
\[ -\dot{j}_0 \]
\[ j_1 \]
\[ -\dot{j}_1 \]
\[ j_2 \]
\[ -\dot{j}_2 \]
\[ \dot{j}_0' \]
\[ -\dot{j}_0' \]
\[ j_1' \]
\[ -\dot{j}_1' \]
\[ j_2' \]
\[ -\dot{j}_2' \]

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The theory of field patterns

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"Effective properties"

"Effective conductivity tensor":

\[
\sigma_\star = \begin{pmatrix} \alpha_\star & 0 \\ 0 & -\beta_\star \end{pmatrix} = \begin{pmatrix} \frac{c_1(c_1+2c_2)(\gamma_1+\gamma_2)}{\gamma_1^2(c_1+c_2)} & 0 \\ 0 & -\frac{(c_1+c_2)[2+(\gamma_2/\gamma_1)]}{c_1(c_1+2c_2)(\gamma_1+\gamma_2)} \end{pmatrix}
\]

"Effective speed":

\[
c_\star = \sqrt{\frac{\alpha_\star}{\beta_\star}} = \frac{c_1(c_1+2c_2)(\gamma_1+\gamma_2)}{c_1+c_2} \sqrt{\frac{1}{\gamma_1(2\gamma_1+\gamma_2)}}
\]

Homogenized equation: \( \nabla \cdot \sigma_\star \nabla V = 0 \)
Numerical results: Transfer Matrix

\[ j(k, m, n + 1) = \sum_{k',m'} T_{(k,m),(k',m')} j(k', m', n) \]

\[ T_{(k,m),(k',m')} = G_{k,k'}(m - m') \]
Periodic solution
Blow up
Eigenvalues of the transfer matrix

The diagram shows the distribution of eigenvalues in the complex plane. The real part of the eigenvalues ranges from -1 to 1, and the imaginary part ranges from -1 to 1. The points are plotted at various positions indicating the complex eigenvalues.
An example of a solution that does not blow up

Periodic, but periodicity greater than that of the field pattern.
Another solution that does not blow up
One more solution that does not blow up
Associated field patterns
Associated field patterns
Associated field patterns

- Associated field patterns of the **first degree**:

\[ W(x, t, \alpha_1, \alpha_2) = \int_{\alpha_1}^{\alpha_2} V(x, t, \alpha) \, d\alpha \]

- Associated field patterns of the **second degree**:

\[ Y(x, t, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}) = \int_{\alpha_{11}}^{\alpha_{12}} d\alpha_1 \int_{\alpha_{21}}^{\alpha_{22}} d\alpha_2 \, W(x, t, \alpha_1, \alpha_2). \]
Part 2: Field patterns in temporal laminates

Main ideas due to Alexander and Natasha Movchan and Hoai Minh Nguyen

Figure: Wave split at temporal interfaces
For $2n + 1$ layers, including $n + 1$ of $\Omega_1$—type and $n$ of $\Omega_2$—type, the “edge wave” coefficient is equal to

$$C_n = \frac{1}{2} \left( 1 + \frac{1}{4} \left( \sqrt{\frac{\alpha_1 \beta_1}{\alpha_2 \beta_2}} + \sqrt{\frac{\alpha_2 \beta_2}{\alpha_1 \beta_1}} - 2 \right) \right)^n,$$

which grows exponentially, as $n \to \infty$ for all cases where the positive coefficients $\alpha$ and $\beta$ are chosen in such a way that $\alpha_1 \beta_1 \neq \alpha_2 \beta_2$. The graphs of $C_n$ for different values of the contrast parameter $\kappa = \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2}$ are shown in the Figure below.
Edge wave amplitude

*Figure:* Edge wave amplitude for different values of the contrast parameter $\kappa$. 
Future work

- Add a small non-linearity
- Add a small imaginary part to $\sigma(x)$
- 2D + time, 3D + time
- Other wave equations
- Effective equation
The million dollar question

Are the fundamental objects in the universe, not particles, not waves, but field patterns?

Thank you for your attention!!
Extending the Theory of Composites to Other Areas of Science

Edited By
Graeme W. Milton

This book changes the landscape of many problems in science, ranging from reformulating Schrödinger’s equation (of importance in Chemistry, Physics, and Material Science) leading to new algorithms for solving multi-electron systems, existing in a new light inverse problems, where one seeks to determine what is inside a body from boundary measurements, that could lead to new methods of imaging, generalizing conservation laws to boundary field equations and inequalities, deriving integral representation formulae for the response of bodies, bounding the transient response of composites, and even introducing a new type of function, called a superfield, where the basic object is a subspace collection, plus many other groundbreaking ideas.

Graeme W. Milton is a distinguished professor of mathematics at the University of Utah, and one of the world leading experts in the field of composites having written one of the definitive texts on the subject, cited "The Theory of Composites". He is a SIAM fellow, and with Iwan and Newall (Milton) was honored for the Society for Industrial and Applied Mathematics (SIAM) Prize for his many and deep contributions to the analysis of composite materials, winner of the Poggendorff Medal of the Society for Experimental Science for "pioneering experimental research of heterogeneous solids", and winner of the Kollros lamp award of the International IUTAM Symposium "for research contributions in the field of composite materials". He has produced over 300 papers in nearly every facet of the sciences. He is known for his work in the theory of composites and metamaterials, but also has made significant contributions to the field of inverse problems and has helped advance the science in this area. The book is presented in collaboration with his students, Marcel Caster, Ornella Mattei, Moti Milgrom, and Aaron Welters, and is a collection of the new form of fractional transverse algorithms for composites.

The work illustrated in this book is authored by Milton as a representation of the work of composite materials.

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14 chapters; 4 coauthored with Maxence Cassier, Ornella Mattei, Moti Milgrom, and Aaron Welters

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Green function for the aligned geometry (1)

\[ j(1, 2, 0) = 1 \quad \Rightarrow \quad \begin{cases} \ G(9, 1, -1) = 1; \ G(10, 1, -1) = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}, \\ \ G(12, 1, -1) = -\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \end{cases} \]

\[ j(2, 2, 0) = 1 \quad \Rightarrow \quad \begin{cases} \ G(1, 2, 0) = -\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \ G(3, 2, 0) = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}, \\ \ G(6, 2, 0) = 1 \end{cases} \]

\[ j(3, 2, 0) = 1 \quad \Rightarrow \quad G(11, 3, -1) = 1 \]

\[ j(4, 2, 0) = 1 \quad \Rightarrow \quad G(8, 4, 0) = 1 \]

\[ j(5, 2, 0) = 1 \quad \Rightarrow \quad \begin{cases} \ G(1, 5, 0) = 1; \ G(4, 5, 0) = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}, \\ \ G(6, 5, 0) = -\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \end{cases} \]
Green function for the aligned geometry (2)

\[ j(6, 2, 0) = 1 \quad \Rightarrow \quad \begin{cases} 
G(7, 6, 0) = -\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \\
G(9, 6, 0) = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \\
G(10, 6, 0) = 1
\end{cases} \]

\[ j(7, 2, 0) = 1 \quad \Rightarrow \quad \begin{cases} 
G(3, 7, 0) = 1; \\
G(4, 7, 0) = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \\
G(6, 7, 0) = -\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}
\end{cases} \]

\[ j(8, 2, 0) = 1 \quad \Rightarrow \quad \begin{cases} 
G(7, 8, 0) = -\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \\
G(9, 8, 0) = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \\
G(12, 8, 0) = 1
\end{cases} \]

\[ j(9, 2, 0) = 1 \quad \Rightarrow \quad G(5, 9, 0) = 1 \]

\[ j(10, 2, 0) = 1 \quad \Rightarrow \quad G(2, 10, 1) = 1 \]

....
Symmetric dynamics for the staggered geometry
Antisymmetric dynamics for the staggered geometry