Fourty most significant publications

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Citation figures are according to Google Scholar as of September 18th 2016

It is always difficult to decide which are your most significant publications, as some oustanding work may be displaced. Here I have chosen those that I feel are the most important while representative of my broad range of interests. These are not necessarily my most highly cited papers, and indeed here I have displaced papers that have been highly cited: specifically I have displaced 6 papers that have been cited 279, 273, 255, 195, 179, and 167 times, respectively, by ones that have been cited 14, 11, 5, 4, 3, and 2 times. I have found that citations sometimes increase dramatically once the significance of the result becomes appreciated [this was especially true of papers (15) and (16) below]. My total citation count of 11,100 ("h-index" of 51) is high for a mathematician.

Monographs:
(1) "The Theory of Composites", Volume 6 of Cambridge monographs on applied and computational mathematics, Cambridge University Press (2002), ISBN 0-521-78125-6 (719 pages): 1886 citations. This book, which took me almost 14 years to write, is now a classic in the field of composite materials. It gives a comprehensive treatment of exact relations of composites, exactly solvable microstructures, approximation schemes, analytic properties of the effective moduli as functions of the component moduli, perturbation expansions for the effective moduli, variational principles for effective moduli and bounds for effective moduli. Many old results are surveyed and many new results are derived. Quoting the review of Allaire on MathSciNet: "Unfortunately, the literature on this topic is scattered over at least over three different communities: mathematics, physics and mechanics, and there existed no book or review paper that would allow a newcomer to get a general knowledge of the state of the art. The book of Graeme Milton fills this gap, and it does so in a splendid manner that will make it the reference book on composite materials for a long time".

(2) "Extending the Theory of Composites to Other Areas of Science", Milton-Patton publishing (2016), ISBN 978-1-48356-919-2 (422 pages). This book changes the landscape of many problems in science, ranging from reformulating Schrödingers equation (of importance to Chemistry, Physics, and Material Science) leading to new algorithms for solving multi-electron systems, casting in a new light inverse problems, where one seeks to determine what is inside a body from boundary measurements, that could lead to new methods of imaging, generalizing conservation laws to boundary field equalities and inequalities, deriving integral representation formulae for the response of bodies, bounding the transient response of composites, and even introducing a new type of function, called a superfunction, where the basic object is a subspace collection, plus many other groundbreaking ideas.

Papers:
(3) "Bounds on the complex permittivity of a two-component composite material", J. Appl. Phys. 52, 5286-5293 (1981): 339 citations. This is the paper for which I received the high-

est number of reprint requests. It was done while I was an undergraduate (doing honors at Sydney University with Professor Ross McPhedran), and the results are now known as the "Bergman-Milton bounds" (type Bergman-Milton bounds in google: Bergman independently, and at the same time, derived some but not all of the results). They bound the response of composites to electromagnetic waves given the response of the two constituent materials. (I wrote 9 other papers in the subsequent year).

(4) "Bounds on the electromagnetic, elastic, and other properties of two-component composites", Phys. Rev. Lett. 46, 542-545 (1981): 308 citations. Bounds that incorporate 3-point correlation function information had been derived by Beran, Molyneux, and McCoy for the effective conductivity, bulk modulus, and shear modulus of composites. For two phase composites this paper notes that the bounds can be reduced to elegant forms that involve the volume fraction of f_1 of phase one and just two parameters ζ_1 and η_1, each taking values between 0 and 1, that can be expressed in terms of the probability that a triangle lands with all vertices in phase 1 when thrown randomly in the composite. These parameters are sometimes referred to as "the Milton parameters".

(5) "The coherent potential approximation is a realizable effective medium theory", Comm. Math. Phys. 99, 463-500 (1985): 155 citations. This gave a rigorous proof that one of the most famous and popular approximation formulas (the coherent potential approximation, also known as the self consistent scheme, or effective medium approximation) for estimating the effective moduli of composites was actually realizable by a limiting microstructure, thus placing it on a sound basis.

(6) "Normalization constraint for variational bounds on fluid permeability", (with J. Berryman), J. Chem. Phys. 83, 754-760 (1985): 85 citations. We noticed that, with suitable choices of trial field, an upper bound on the permeability of a porous medium implied by variational principles of Prager could be arbitarily close to zero no matter what the geometry of the medium. If this was true, then one could never have coffee in the morning and one could never extract oil from a porous rock containing it. Consequently we found an error in the variational principle: Prager had wrongly, and without proper analysis, guessed the normalization term entering the expression involving the effective fluid permeability in the variational principle. When we corrected the variational principle the upper bound in porous media was no longer close to zero, and one could happily drink coffee and extract oil again.

(7) "A proof that laminates generate all possible effective conductivity functions of two-dimensional, two-phase materials", Advances in Multiphase Flow and Related Problems, 136-146, ed. by G. Papanicolaou (SIAM, Philadelphia, 1986): 11 citations. In two-dimensional composites of two isotropic phases the effective conductivity is a matrix valued function of the two component conductivities that has a Stieltjes type integral representation involving a positve semidefinite matrix valued measure. Additionally this function satisfies a constraint known as the Keller-Dykhne-Mendelson phase interchange relation. Here we proved that these constraints completely characterize the possible conductivity functions. Specifically, given any 2×2 symmetric matrix valued rational function satisfying these constraints one can identify an hierarchical laminate material that has this function as its effective conductivity for all real or complex values of the component conductivities.

The proof is by induction on the degree of the rational function. Thus, in this context, hierarchical laminates form a set of representative composites.

(8) "Multicomponent composites, impedance networks, and new types of continued fraction I" , Comm. Math. Phys. 111, 281-327 (1987): 61 citations. A function of a single complex variable can be easily expanded as a continued fraction in various ways. Truncations of certain continued fraction expansions play an important role in obtaining bounds on Stieltjes and Herglotz functions that utilize limited information about the function, such as series expansion coefficients up to a given order and known values of the function at a set of points. These correspond to bounds on the effective moduli of two-phase composites, as functions of the two component conductivities. By homogeneity of this conductivity function, and without loss of generality, one can set one of the component conductivities to unity. For multiphase composites one can decompose the relevant Hilbert space of fields into a hierarchy of subspaces. Truncating the subspace decomposition at a given level then generates bounds on the effective tensor that incorporate information in normalization and weight matrices that can be extracted from the series expansions for the fields. These bounds as functions of the component conductivites in turn can be represented as a new type of continued fraction incorporating matrices of increasing dimension as one progresses down the continued fraction.

(9) "Asymptotic studies of closely spaced, highly conducting cylinders ", (with R.C. McPhedran and L. Poladian) Proc. Roy. Soc. Lond. A 415, 185-196 (1988): 104 citations. At large volume fractions the effective conductivity of randomly or periodically placed disks or aligned cylinders of highly conducting material is controlled by the conductivity across the gaps between neighboring cylinders. The extreme case where the conductivity of the disks is infinite and the disks are almost touching, and the other extreme case where the disks are touching and the conductivity of the disks tends to infinity had been studied before. However no satisfactory approximation had been obtained that interpolates between these two extreme cases and that is uniformly valid for highly (not infinitely) conducting disks close to contact (not in contact). Here such an approximation was derived: the key idea was to replace the discrete distribution of image dipoles by a continuous distribution and then use a simple solution to the functional equation satisfied by this distribution. The approximation was in excellent agreement with numerical results.

(10) "Composite materials with Poissons ratio close to -1, Journal of the Mechanics and Physics of Solids", 40, 1105-1137 (1992): 276 citations. It had long been questioned as to whether materials with a negative Poisson ratio could exist: such materials become fatter as they are stretched. My paper was the first proof that isotropic elastic composite materials, without sliding surfaces, can exhibit a negative Poisson ratio, thus placing Lakes 1987 experimental results on a sound footing.

(11) "Invariant properties of the stress in plane elasticity and equivalence classes of composites", (with A.Cherkaev and K.A.Lurie) Proc. Roy. Soc. Lond. A, 438, 519-529 (1992): 120 citations. This paper proved that when the local elastic moduli of an inhomogeneous plate undergo an appropriate shift, then the the effective elastic moduli undergo exactly the same shift. This has become known as the CLM theorem (type CLM theorem in google scholar), and has had many generalizations.

(12) "On the effective viscoelastic moduli of two-phase media: I Rigorous bounds on the complex bulk modulus", (with L.V.Gibiansky) Proc. Roy. Soc. Lond. A, 440, 163-188 (1993): 80 citations Here viscoelastic minimization variational principles due to Cherkaev and Gibiansky were employed to obtain bounds on the quasistatic complex bulk modulus of two phase mixtures at a given frequency, given the volume fractions of the phases and the complex bulk moduli of the phases. The bounds have a beautiful interpretation as a lens shaped region in the complex bulk modulus plane, and geometries were identified that have effective bulk moduli on the boundary of the lens. Significantly, and for the first time, the "Y-tensor" was given a simple geometric interpretation in two-phase composites. Later bounds were obtained with J.Berryman on the complex shear modulus of composites.

(13,14) The companion papers, "Sets of conductivity and elasticity tensors stable under lamination", (with G.A.Francfort), Comm. Pure Appl. Math., 47, 257-279 (1994), 30 citations, and "A link between sets of tensors stable under lamination and quasiconvexity", Comm. Pure Appl. Math., 47, 959-1003 (1994), 17 citations, prove that "G-closures", i.e. the sets of all possible effective effective tensors of composites built say from n-phases with prescribed elasticity tensors (possibly in prescribed proportions) can be characterized by minimums of energies and complementary energies in much the same way that convex sets can be characterized by their Legendre transforms. What gives the "G-closure" sufficient convexity to be characterized in this way is the property that if we take two materials in the "G-closure" and laminate them together then the mixture must have an effective tensor that lies in the "G-closure". (It had earlier been suggested by Cherkaev and Gibiansky that bounding sums of energies and complementary energies would be useful for bounding the "G-closure" but they did not establish that the "G-closure" could always be characterized in this way". As emphasized in the paper "On the possible effective elasticity tensors of 2-dimensional and 3-dimensional printed materials" (with Marc Briane and Davit Harutyunyan), Memocs (submitted, see arXiv:1606.03305 [cond-mat.mtrl-sci]) this defines a new type of transform that generalizes the Legendre transform.

(15) "Optical and dielectric properties of partially resonant composites" (with N.A.Nicorovici and R.C.McPhedran) Phys. Rev. B 49, 8479-8482 (1994): 201 citations. This paper was ahead of its time and obtained some striking results, showing that in quastistatics (when the wavelength is much bigger than the inclusion) an annular shell with negative moduli could magnify the core and thus be invisible when the core was empty. If this equivalence held, and the core was non-empty, it implies by the method of images that when an appropriately placed dipole source is outside one would expect an image source outside (!) the annular shell (in the physical region). We explored this and found that indeed when the loss in the shell tended to zero the field on one side of the "expected image source" looks like there is an actual point source there, while on the other side of the "expected image source" we found the field exhibited oscillations (the first discovery of "anomalous localized resonance", where the region where the field blows up depending on the location of the source). Anomalous resonance and "virtual image sources" are the mechanisms which are responsible for the very similar phenomena of perfect imaging by superlenses discovered in 2000 by Pendry (and which has now received about 10,000 citations). Our work preceeded it by 6 years.

(16) "Which elasticity tensors are realizable?" (with A.V.Cherkaev) ASME J. Eng. Mat.

Tech. 117, 483-493 (1995): 131 citations. In three-dimensional elastostatics the elasticity tensor of a stable material is represented by a positive definite 6×6 symmetric matrix (technically a 4th order tensor). This paper addressed the question of whether there exist microstructures, built from a very stiff material and a very compliant material, which could realize every positive definite symmetric matrix (in the 21-dimensional space in which these matrices live). We found the answer was yes, and the key to the proof was finding extremal materials which were called pentamode materials. They are a bit like fluids: fluids only support one loading (bulk compression), pentamodes also only support one loading (but that loading may be an arbitrary combination of bulk compression and shear). We never dreamt that our pentamodes would be made, but the group of Martin Wegener did just that in 2012, 19 years after our work, and subsequently there has been a lot of work on pentamodes: (type pentamode in google). In 2012, before they built pentamodes the group of Wegener in the paper, "Tailored 3D Mechanical Metamaterials Made by Dip-in Direct-Laser-Writing Optical Lithography" wrote: "In addition, the "holy grail" of mechanical materials, namely pentamode materials, that can be seen as the mother of all materials, might become accessible as well. Pentamodes, suggested by Milton and Cherkaev in 1995, are solids that behave like fluids with a very small effective shear modulus."

(17) "Finite frequency range Kramers–Kronig relations: Bounds on the dispersion" (with D.J.Eyre and J.V.Mantese) Phys. Rev. Lett., 79, 3062- 3064 (1997): 80 citations. The Kramers–Kronig relations are famous because (for example) they enable experimentalists to determine the real part of the dielectric constant of a material (which in a sense measures how it bends light) as a function of frequency from measurements of the imaginary part of the dielectric constant (which in a sense measures how it absorbs light) as a function of frequency. However they require measurements over the entire frequency range (from zero frequency to infinite frequency) which is never the case in practice. By contrast, our bounds correlate measurements within a fixed frequency window, and thus can be used as a rigorous tool for checking the consistency of experimental measurements.

(18) "A fast numerical scheme for computing the response of composites using grid refinement" (with D.J.Eyre) Eur. Physical J. (Applied Physics), 6, 41-47 (1999): 153 citations. This showed that (for linear materials) the Fast Fourier Transform method of Moulinec and Suquet (1994, 276 citations) for computing the effective properties of composites could be vastly accelerated, and our method remains one of the most efficient Fast Fourier transform methods for solving for the fields and moduli of linear composites (Moulinec and Silva, 2014). It has recently been very sucessfully generalized to nonlinear materials by Willot.

(19) "Exact relations for effective tensors of composites: necessary and sufficient conditions" (with Y.Grabovsky and D.Sage) Comm. Pure. Appl. Math., 53, 300-353 (2000): 46 citations. Although this paper has received relatively few citations (perhaps because few people understand it) I consider it a major achievement. Many, some famous, scientist derived, exact microstructure independent relations satisfied by the effective moduli of composites, and microstructure independent links between effective tensors (which are very useful as benchmarks for testing numerics and approximation theories) usually one at a time. These scientists included Wood, Hill, Helsing, Movchan, Cribb, Rosen and Hashin, Levin, Keller, Dykhne, Dvorak, Murat and Tartar, Shklovskii, Benveniste, Milgrom and Shtrikman, Bergman, Strelniker, Lurie, Cherkaev, Berdichevskii, Schulgasser, Gassmann,

Berryman, Norris, Straley, Chen, and myself. In this paper we developed a very general theory, encompassing all these results, which reduced the search for exact relations to an algebraic one: subsequently Grabovsky has used this powerful theory to systematically obtain an enormous number of new exact relations.

(20) "Proof of a conjecture on the conductivity of checkerboards", J. Math. Phys., 42, 4873–4882 (2001): 30 citations. In 1985 Mortola and Steffé conjectured a formula for the effective conductivity tensor of a checkerboard structure where the unit cell of periodicity is square and subdivided into four equal squares each having a different conductivity. This paper proves their conjecture using various symmetry relations. At the same time the conjecture was proved independently using an entirely different complex analysis method by Craster and Obnosov.

(21) "Neutral coated inclusions in conductivity and anti-plane elasticity" (with S.K.Serkov), Proc. Roy. Soc. Lond. A, 457, 1973-1997 (2001): 49 citations. Here we determine a variety of shapes of two-dimensional inclusions consisting of an isotropic conducting core surrounded by an isotropic conducting shell inserted in a possibly anisotropic conducting matrix such that the electric field outsde the inclusion is completely uniform. (The inclusion is thus invisible to this applied field). The free boundary problem has an elegant solution using conformal mapping techniques.

(22) "Exact band structure for the scalar wave equation with periodic complex moduli", Physica B, 338, 186-189 (2003): 2 citations. Here we show that if a medium is such that the Fourier transform of the complex coefficients entering the wave equation vanishes in half of Fourier space, then one can exactly determine the band structure. In fact it is the same as for a homogeneous medium.

(23) "On the cloaking effects associated with anomalous localized resonance" (with N.A.Nicorovici), Proc. R. Soc. A, 462, 3027-3059 (2006): 445 citations. Here we found and proved a spectacular result: any finite collection of polarizable dipoles (or dipolar sources radiating finite energy) sufficiently near a superlens would become invisible to any external field in the limit as the loss in the lens tends to zero. As can be seen by googling "anomalous localized resonance", this generated considerable attention (along with the work of Leonhardt, and Pendry, Schuring and Smith, whose work on "transformation optics" type cloaking appeared slightly after ours).

(24) "On cloaking for elasticity and physical equations with a transformation invariant form" (with M.Briane and J.R.Willis). New J. Phys. 8, 248 (2006): 650 citations. Here we gave an easy to follow derivation of the known result, central to "transformation optics", that Maxwells equations in inhomogeneous media are invariant under coordinate transformations, and we showed that the equations of continuum elastodynamics are not form invariant, but instead the more general "Willis equations" are invariant (these were derived by John Willis by ensemble averaging the elastodynamic behaviour in composites, and are analogous to the bianisotropic equations of electromagnetism). We devised simple to analyse models (of materials with masses and springs in cavities) exhibiting anisotropic and possibly complex "effective mass density" over a range of frequencies.

(25) "Quasistatic cloaking of two-dimensional polarizable discrete systems by anomalous res-

onance" (with N.A.Nicorovici, R.C.McPhedran, and L.C.Botten) Optics Express 15, 6314-6323 (2007). This was the top Optics research paper in terms of downloads for 2007 with over 13,000 downloads: 159 citations. Here we obtained beautiful numerical simulations confirming the theory developed in paper (11), and showed that cloaking also works for quadrupoles, and even "objects" in the shape of the Aston Martin of James Bond.

(26) "Non-magnetic cloak with minimized scattering" (with W.Cai, U.K.Chettiar, A.V.Kildishev, and V.M.Shalaev), Appl. Phys. Lett. 91, 111105 (2007): 332 citations. At a fixed frequency the ideal cloak as given by the prescription of Pendry, Schurig, and Smith, has a magnetic permeability equal to the electrical permittivity. In geometric optics the trajectory that light follows is controlled by the refractive index, which is the square root of the product of magnetic permeability and electrical permittivity. Thus, as noticed by Schurig et.al. (2006), one can adjust the moduli to more experimentally desired values, while keeping the geometric optics trajectory fixed. In fact, in this way, Cai, et.al. (2007) found that one could get an approximate cylindrical cloak for transverse magnetic (TM waves) built entirely from non-magnetic materials. These approaches have a disadvantage: there is an impedance mismatch at the boundary of the cloak that causes significant reflection. In this paper we note that the impedance mismatch can be eliminated by using a smoother transformation in the design, and our numerical results show a significant reduction in scattering compared to the earlier approximate cloaks.

(27) "Opaque perfect lenses" (with N.A.Nicorovici and R.C.McPhedran), Physica B 394, 171-175 (2007): 21 citations. Here we showed that Sir John Pendry's most cited paper "Negative refraction makes a perfect lens", with over 10,000 citations according to Google Scholar (and associated with the co-award to John Pendry of the million dollar Kavli prize), is fundamentally flawed. Contrary to Pendry's claim that the lens is perfect, it actually has zero transmission when a dipole power source radiating fixed (finite) energy is close to the lens (less than a distance $d/2$ when the lens has a thickness d). Strikingly, in the long time limit, no energy propogates in the direction away from the lens either –the source becomes cloaked.

(28) "On modifications of Newton's second law and linear continuum elastodynamics" (with J.R.Willis), Proc. R. Soc. A, 463, 855-880 (2007): 278 citations. In this paper, we suggested a new perspective, where Newton's second law of motion is replaced by a more general law which is a better approximation for describing the motion of bodies with microstructure. The relation between the force and the acceleration is non-local (but causal) in time. Conversely, for every response function satisfying these properties, and having the appropriate high-frequency limit, there is a model which realizes that response function. In many circumstances, the differences between Newton's second law and the new law are small, but there are circumstances, such as in specially designed composite materials, where the difference is enormous.

(29) "Solutions to the Pólya-Szegö conjecture and the weak Eshelby conjecture" (with H. Kang). Arch. Rat. Mech. Anal. 188, 93-116 (2008): 59 citations. This paper resolved an outstanding conjecture of Eshelby that had been the focus of much attention. In 1961 Eshelby made the famous conjecture that an inclusion must be of ellipsoidal shape if for any uniform elastic loading at infinity the strain field inside the inclusion is uniform. We

proved this conjecture in three dimensions. Also we proved the oustanding 1951 conjecture of Pólya and Szegö that the inclusion whose polarization tensor has minimal trace for a given volume would take the shape of a sphere.

(30) "Minimization variational principles for acoustics, elastodynamics, and electromagnetism in lossy inhomogeneous bodies at fixed frequency" (with P.Seppecher and G.Bouchitte), Proc. R. Soc A, 465, 367-396 (2008): 14 citations. Energy minimization principles have been known since the time of Dirichlet and Thompson and have had a huge number of applications. Here we extended these famous variational principles to wave propogation at fixed frequency in inhomogeneous (slightly absorbing) bodies, or alternatively to complex frequency. They are power, rather than energy, minimization principles. The variational principles should have many applications to bounding the response of such bodies to applied fields, and used in an inverse fashion should yield information about what is inside the body.

(31) "Solutions in folded geometries, and associated cloaking due to anomalous resonance" (with N.A.Nicorovici, R.C.McPhedran, K.Cherednichenko, and Z.Jacob), New J. Physics, 10, 115021 (2008): 79 citations Here we show how solutions in abstract geometries, such as a shell geometry where the core radius is bigger than the shell radius, can be given a physical meaning, through a "transformation optics" unfolding of the geometry, transforming it to one where the shell radius is outside the core radius and the material in the shell is anisotropic. This differs from the unfolding transformation suggested by Leonhardt and Philbin, in that in the folded geometry the field takes three different values on the three different sheets.

(32) "Homogenization of the three-dimensional Hall effect and change of sign of the Hall-coefficient" (with M.Briane) Arch. Rat. Mech. Anal. 193, 715-736 (2009): 22 citations. It is common to read in elementary Physics textbooks that in classical physics the sign of the Hall coeficient determines the sign of the charge carrier, based on the direction of the force by an electron (or hole) moving in a magnetic field. Here we proved that this folklore is wrong: by combining 3 materials each with positive Hall coefficient we proved that one could get a composite with negative Hall coefficient. This was verified numerically by the group of Martin Wegener, who simplified the design to one material plus void, and just recently they have experimentally constructed "the world's first Hall-effect reversal material".

(33) "Active exterior cloaking for the 2D Laplace and Helmholtz equations" (with F.Guevara Vasquez and D.Onofrei) Phys. Rev. Lett. 103, 073901 (2009): 91 citations. Here we developed a new method for cloaking, which has the important advantage that it is broadband, provided one knows the incoming signal. It is exterior in the sense that the cloaking region lies outside the cloaking region. For a given incoming wave a small number of sources that do not radiate into the far field create a quiet zone within which the object to be cloaked can be placed. It received considerable attention (type Active exterior cloaking in google).

(34) "Electromagnetic circuits" (with P.Seppecher), Networks and Heterogeneous Media 5, 335–360 (2010): 5 citations. We are all familiar with models composed of masses and springs and these can be seen as a discrete approximation to linear elasticity. They can also be viewed as an approximation to a model where one has a network of cylinders of a massless elastic medium, joined to nodes containing a rigid material having mass, with everything being surrounded by void having zero density and zero elastic moduli. This paper, for

the first time, gives an electromagnetic analog of a mass-spring system. The objects that are glued together to make the electromagentic circuit are thin triangles of a material with high magnetic permeability and zero electric permittivity (or vice-versa) joined by cylinders with high electrical permittivity and zero magnetic permeability (or vice-versa) with the system being surrounded by a medium having infinite magentic permeabilty and zero electric permittivity (or vice-versa). These can only be realized approximately since zero magnetic permeability or zero electric permittivity can only (almost) be achieved at one frequency not over a range of frequencies.

(35) "Complete characterization and synthesis of the response function of elastodynamic networks" (with F.Guevara Vasquez and D.Onofrei) J. Elasticity 102, 31–54 (2011): 3 citations. Despite the few citations this result represents the natural generalization to multiterminal networks of springs and masses of Foster's (1924) classical theorem on the response of two terminal passive electrical network (important in network synthesis). Our paper shows exactly what sort of responses (time dependent movement of the terminal nodes in response to time dependent forces acting at these modes) can be realized. (Curiously, Foster's theorem hasn't been extended to multiterminal electrical networks).

(36) "Complete characterization of the macroscopic deformations of periodic unimode metamaterials of rigid bars and pivots". J. Mech. Phys. Solids 61, 1543–1560 (2013): 24 citations. Around the time of the steam engine there was great interest in mechanisms that could convert straight line motion (of the piston) into circular motion of a wheel. Kempe is his famous 1876 univerality theorem showed that from straight line motion one could obtain any desired motion (more specifically, any portion of an algebraic curve). My paper in a sense extends his result to periodic materials constructed from rigid shapes linked by hinges (which can be approximated by narrow necks of flexible material): one can find hierarchical materials whose macroscopic response is confined to track any desired trajectory in the 6-dimensional space of (affine) macroscopic deformations. A particular example are materials whose deformation is a dilation over a finite range of deformation: these can change their size but not (easily) their shape. Martin Wegener's group, with minor assistance from me, constructed an approximation to such materials.

(37) "Sharp inequalities which generalize the divergence theorem: an extension of the notion of quasi-convexity", Proc. Roy. Soc. A 469, 20130075 (2013): 4 citations. See also the addendum: Proc. Roy. Soc. A 471,20140886 (2015). Here we consider integrals over a body of certain quadratic functions of fields that satisfy some differential constraints, such as being the gradient of a potential. If these quadratic functions satisfy a condition we call Q^*-convexity, that generalizes quasiconvexity, then a sharp lower bound on the integral can be obtained and this lower bound is expressed entirely in terms of boundary fields. The most interesting Q^*-convex functions are the extremal ones: these lose their Q^*-convexity whenever a convex quadratic function is subtracted from them. I give an algorithm for constructing extremal Q^*-convex functions. It is anticipated that these inequalities may prove to be very useful in analysis.

(38) "Rigorous bounds on the effective moduli of composites and inhomogeneous bodies with negative-stiffness phases" (with D.Kochmann), J. Mech. Phys. Solids 71, 46-63 (2014): 15 citations. In work that attracted a lot of attention, it had been suggested by Lakes and

Drugan that one might be able to combine a phase with positive stiffness with a phase with negative stiffness to obtain a composite with a very large static stiffness: for example, if one uses a harmonic average formula for the effective stiffness then the result can be infinitely large at a suitable volume fraction. This paper rules out such unusual behavior in stable composites by pointing out that many of the standard variational principles and associated bounds still hold for stable composites even when some of the constituent phases have negative stiffness. Thus the static effective bulk modulus can never be greater than the bulk modulus of the consituents even when some of these are negative.

(39) "On the possible effective elasticity tensors of 2-dimensional and 3-dimensional printed materials" (submitted, arXiv:1606.03305 [cond-mat.mtrl-sci]). A natural question to ask is: what elasticity tensors can be obtained by mixing, in fixed proportions, one phase with a known elasticity tensor with a second phase which is void, or almost void? Three dimensional elasticity tensors have 21 elements, but as we are free to make rotations, we can rotate the material to a reference frame where 3 elements are zero, leaving 18 independent elements. Even a distorted hypercube in 18 dimensions requires about 4.7 million parameters to specify it, so characterizing the set of possible elasticity tensors is no easy task. The set is described by seven different functions, each representing the minimum as the microstructure is varied of a sum of a set of energies associated with p applied strains plus a set of complementary energies associated with $6 - p$ applied stresses. In the case where $p \geq 3$, or when $p = 1$ or 2 and some ancillary conditions are satisfied, we provide a prescription for exactly calculating these functions, and we identify limiting geometries that come arbritraily close to minimizing the sum. (The case where $p = 6$ is trivial since the minimum is clearly zero, being attained for a geometry consisting of islands surrounded by void, and the case where $p = 0$ was treated by Avellaneda back in 1987).

(40) "Analytic and Polynomial Materials" (submitted). Here new classes of inhomogeneous bodies are introduced for which on can easily obtain exact solutions for the fields inside the body. The set of partial differential equations satisfied by the fields are replaced by a set of ordinary differential equations.