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Transformation Optics Discovered
О ВОЗМОЖНОСТИ СОПОСТАВЛЕНИЯ ТРЕХМЕРНЫХ ЭЛЕКТРОМАГНИТНЫХ СИСТЕМ С НЕОДНОРОДНЫМ АНИЗОТРОПНЫМ ЗАПОЛНЕНИЕМ

Л. С. Долин

Показано, что, основываясь на инвариантности уравнений Максвелла, относительно определенного вида преобразования метрики пространства и проницаемостей среды, можно исследовать трехмерные системы с неоднородным анизотропным заполнением путем их сопоставления с другими, более простыми трехмерными системами.

TO THE POSSIBILITY OF COMPARISON OF THREE-DIMENSIONAL ELECTROMAGNETIC SYSTEMS WITH NONUNIFORM ANISOTROPIC FILLING

L. S. Dolin

It was shown that it is possible to investigate three-dimensional systems with nonuniform anisotropic filling by comparison them with other, more simple three-dimensional systems. The examination is made basing on an invariance of Maxwell's equations relative to the certain type of transformation of space metric and medium permeability and permittivity.
It is known that, for certain systems electromagnetic fields can be found without directly solving the Maxwell’s equations, by comparing them with the fields for other, simpler, systems. Here one can mention two-dimensional static systems, which can be investigated using the method of conformal mapping of boundaries, three-dimensional static systems with piecewise constant dielectric (magnetic) material parameters for domains bounded by equipotential surfaces; examples also include some high-frequency systems with optimally tuned distributions of the permittivity/permeability. Indeed, for some systems with a homogeneous medium it is possible to select a metric in which Maxwell’s equations are formally no different from the equations describing the field in a corresponding inhomogeneous medium, but expressed in different generalized coordinates. In this way, one can find particular solutions of some two-dimensional problems and of problems having axial symmetry [1-3].
The aim of the present communication is a comparative analysis of analogous three-dimensional electromagnetic systems.

Let us write down the Maxwell’s equations in generalized orthogonal coordinates:

\[
\frac{1}{h_i h_j} \left[ \frac{\partial}{\partial \xi^i} (h_j H_j) - \frac{\partial}{\partial \xi^j} (h_i H_i) \right] = \frac{4\pi}{c} j^e_k + \frac{1}{c} \frac{\partial}{\partial t} \sum_{l=1}^{3} \epsilon_{kl} F_l ;
\]

\[
\frac{1}{h_i h_j} \left[ \frac{\partial}{\partial \xi^i} (h_j E_j) - \frac{\partial}{\partial \xi^j} (h_i E_i) \right] = -\frac{4\pi}{c} j^m_k - \frac{1}{c} \frac{\partial}{\partial t} \sum_{l=1}^{3} \nu_{kl} H_l ;
\]

\[
\frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial \xi^i} \left( h_j h_k \sum_{l=1}^{3} \epsilon_{il} E_l \right) = 4\pi \rho^e ;
\]

\[
\frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial \xi^i} \left( h_j h_k \sum_{l=1}^{3} \nu_{il} H_l \right) = 4\pi \rho^m .
\]
Here $E$ and $H$ are the electric and magnetic fields strengths, $j^e$ and $j^m$ are densities of the electric and magnetic currents, $\rho^e$ and $\rho^m$ are volumetric densities of corresponding charges, $\varepsilon_{ik}$ and $\mu_{ik}$ are the components of the tensors of dielectric permittivity $||\varepsilon_{ik}||$ and of magnetic permeability $||\mu_{ik}||$ respectively, the $\xi_i$ are generalized orthogonal coordinates ($i=1,2,3$) having the dimensions of length, the $h_i$ are the non-dimensional metric Lame coefficients (scaling factors) corresponding to these coordinates; the indices $i,j,k$ form all possible cyclic permutations of 1, 2, 3.
It is straightforward to verify that the system (1) of equations for the covariant components of the field vectors is invariant with respect to such transformations of the metric of the space, the medium’s permittivity and permeability and of the distributions of the sources of the electromagnetic field, for which the following entities are simultaneously conserved:

$$
\varepsilon_{kl} \frac{h_i h_j}{h_l} = \text{invar}, \quad i^e h_j h_k = \text{invar}, \quad \varphi^e h_1 h_2 h_3 = \text{invar},
$$

$$
\varphi_{kl} \frac{h_i h_j}{h_l} = \text{invar}, \quad j^m h_j h_k = \text{invar}, \quad \rho^m h_1 h_2 h_3 = \text{invar},
$$

where $$k, l = 1, 2, 3$$ and $$i \neq j \neq k$$. The invariance of the equations implies the possibility of existence of invariant solutions*:

$$
h_i E_i = \text{invar}; \quad h_i H_i = \text{invar}.
$$

(See the above equations)
Conditions (2), (3) can form the foundation for a principle of comparison of fields in three-dimensional systems with an inhomogeneous anisotropic filling. In general, the anisotropy is inevitable, due to the constraints (2) on \( \varepsilon \) and \( \mu \).

Let us illustrate the generic conditions (2) and (3) on several simple examples.

Let us start with problems of oscillations of resonators with ideal boundaries. It follows from the obtained formulas that, if one makes a formal change of variables in the expressions for the fields in a resonator, then the functions obtained by such a transformation can be regarded (up to a functional factor) as components of the electromagnetic field in the new resonator, different from the initial one by its shape as well as by the parameters of the medium which fills it.
Let us take, for example, as the reference system, a resonator whose boundaries are formed by intersection of the coordinate surfaces of a cylindrical system (Fig. 1), and let the normalized permittivity/ permeability $\varepsilon$ and $\mu$ be equal to 1. Changing from the cylindrical coordinates $r, \phi, z$ to Cartesian coordinates $x, y, z$, we obtain a rectangular resonator filled by the medium with the permittivity/ permeability

$$\| \varepsilon_{ik} \| = \| \mu_{ik} \| = \begin{vmatrix} x & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & x \end{vmatrix}.$$
The spectrum of the eigenfrequencies of this resonator is the same as for the original resonator, and the eigenfunctions can be found in the following way: if \( \Phi_1(r, \phi, z) \), \( \Phi_2(r, \phi, z) \), \( \Phi_3(r, \phi, z) \) are components of the electric and magnetic field of the original resonator along the axes of the cylindrical system, the projections of the field of the rectangular resonator on the Cartesian directions equal \( \Phi_1(x, y, z) \), \( x^{-1}\Phi_2(x, y, z) \), \( \Phi_3(x, y, z) \).

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* Analogous relations can be obtained also for non-orthogonal coordinate systems.
Unfortunately, the conditions (2), (3) do not allow one to transform the boundaries and the properties of the medium independently. Setting a rule for transforming one of the parameters of the system (shape of the boundary, $|\varepsilon_{ik}|$ or $|\mu_{ik}|$) uniquely determines all the others, which naturally restricts the potential of the present method.

If a transformation of coordinates changes the metric within a bounded region, the parameters of the medium and the expressions for the fields outside this region are preserved. Such transformations can be used for constructing non-reflecting inhomogeneities.
Let us obtain, for example, the parameters of a system which would emerge as a result of a transformation of free space when changing spherical coordinates \( r, \theta, \phi \) into coordinates \( R(r), \theta, \phi \), where \( R= R(r) \) is a function of the radial coordinate satisfying the condition

\[ R(r) \to r. \] (4)

The scale factors for this system are equal to \( h_R = dr(R)/dR \), \( h_\theta = r(R) \), \( h_\phi = r(R) \sin \theta \), and for the permittivity/ permeability of the medium we obtain the following formulas:

\[ (\text{See equation above}) \] (5)

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\[ (\text{See equation above}) \] (5)
As one can see, under the condition (4) $\varepsilon_{lk} = \mu_{lk} \rightarrow 1$ as $r \rightarrow \infty$. A plane wave incident from infinity upon an inhomogeneity with parameters (5) will go through it without distortions.

Let us give an example of the correspondence between systems with sources. Let the reference system be free space, in which a charged particle uniformly moves along a circle of a radius $a$. Let us take as coordinates of the original system $\ln(r/a), \phi, z$ (where $r, \phi, z$ are cylindrical coordinates) and let us consider how the pattern of the radiation and properties of the system changes if we switch to Cartesian coordinates. Let the coordinate $\phi$ vary within $(-\infty, +\infty)$. Then, in the system $x, y, z$, we obtain an infinite sequence of particles moving along the $y$-axis a distance $2\pi$ apart. Applying the relations (2) shows that the magnitude of the charge and its velocity remain the same but the space becomes filled with a medium with the following permittivity/ permeability tensors:
As one can see, the radiation is no longer the result of a non-uniformity of the charge’s motion, but is caused by an interaction with the medium: it can be interpreted as Cherenkov’s radiation in an anisotropic medium.

The above examples, and many more can be devised, show that the correspondence method based on relations (2), (3), can be useful for solving a number of electromagnetic problems. At the same time it has to be emphasized that the complexity
of the medium’s parameters which emerge as a result of the method’s application – non-uniformity, anisotropy, presence of zeros and poles – considerably limits its applicability.

In conclusion, let us perform a limiting procedure for a two-dimensional transformation and generalize some of the results obtained with its help in [1-3]. If one of the two compared systems is considered in Cartesian coordinates $x, y, z$ and the other in curvilinear coordinates $u, v, z$, of which the former two coordinates appear as a result of a conformal mapping of the (first two) Cartesian coordinates, and the third coordinate ($z$) is Cartesian, the formulas (2) transform for diagonal tensors into the relations:

$$
\varepsilon^{(2)}_{kk} = \varepsilon^{(1)}_{kk}; \quad \mu^{(2)}_{kk} = \mu^{(1)}_{kk}, \quad (k = 1, 2);
$$

$$
\varepsilon^{(2)}_{33} = \varepsilon^{(1)}_{33} / h^2; \quad \mu^{(2)}_{33} = \mu^{(1)}_{33} / h^2,
$$
where the upper index “1” denotes entities of the first system, while the index “2” corresponds to those of the second system, and $h$ is the scale factor for coordinates $u, v$.

For two-dimensional fields, the tensors can be replaced by scalars and formulas (6) will transform into known relations:

\[
\begin{align*}
\text{for } & H_z = 0, \quad E_z \neq 0 & \varepsilon^{(2)} = \varepsilon^{(1)}/h^2, \\
\text{for } & E_z = 0, \quad H_z \neq 0 & \mu^{(2)} = \mu^{(1)}/h^2.
\end{align*}
\]
Formulas (6) allow one to easily generalise to the three-dimensional case the results in [1-3] devoted to questions of correspondence of waveguides with the aid of inhomogeneous media. For example, to eliminate a reflection in a waveguide having a rectangular cross-section in any of the planes containing the z-axis and arbitrary cross-section in the \( z = \text{const} \) (Fig. 2), it is necessary to fill it with an anisotropic medium with tensor (6), where \( h \) should be understood as a metric coefficient of such a coordinate system in which the waveguide's boundaries coincide with lines of constant value of one of these coordinates. For waves of TE type one can set \( \mu=1 \) and ensure the desired correspondence by appropriately choosing the component \( \varepsilon_{33} \).
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Translated anonymously by some exceptional mathematicians fluent in both English and Russian.

Graeme Milton, August 3rd, 2016