NOTES ON RIEMANN'S ZETA FUNCTION

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1. Gamma function

1.1. **Definition of the Gamma function.** The integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

is well-defined and defines a holomorphic function in the right half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. This function is Euler's *Gamma* function.

First, by integration by parts

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = -t^z e^{-t}\Big|_0^\infty + z \int_0^\infty t^{z-1} e^{-t}\, dt = z\Gamma(z)$$

for any z in the right half-plane. In particular, for any positive integer n, we have

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!\Gamma(1).$$

On the other hand,

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 1;$$

and we have the following result.

1.1.1**. Lemma.**

$$\Gamma(n) = (n-1)!$$

for any $n \in \mathbb{Z}$.

Therefore, we can view the Gamma function as a extension of the factorial.

1.2. **Meromorphic continuation.** Now we want to show that Γ extends to a meromorphic function in $\mathbb{C}$. We start with a technical lemma.

1.2.1**. Lemma.** *Let c_n, $n \in \mathbb{Z}_+$, be complex numbers such such that $\sum_{n=0}^\infty |c_n|$ converges. Let*

$$S = \{-n \mid n \in \mathbb{Z}_+ \text{ and } c_n \neq 0\}.$$

Then

$$f(z) = \sum_{n=0}^\infty \frac{c_n}{z+n}$$

converges absolutely for $z \in \mathbb{C} - S$ and uniformly on bounded subsets of $\mathbb{C} - S$. The function f is a meromorphic function on $\mathbb{C}$ with simple poles at the points in S and $\operatorname{Res}(f, -n) = c_n$ for any $-n \in S$.

Proof. Clearly, if $|z| < R$, we have $|z+n| \geq |n-R|$ for all $n \geq R$. Therefore, we have $|\frac{1}{z+n}| \leq \frac{1}{n-R}$ for $|z| < R$ and $n \geq R$. It follows that for $n_0 > R$, we have

$$\left|\sum_{n=n_0}^{\infty} \frac{c_n}{z+n}\right| \leq \sum_{n=n_0}^{\infty} \frac{|c_n|}{|z+n|} \leq \sum_{n=n_0}^{\infty} \frac{|c_n|}{n-R} \leq \frac{1}{n_0-R} \sum_{n=n_0}^{\infty} |c_n|.$$

Hence, the series $\sum_{n>R} \frac{c_n}{z+n}$ converges absolutely and uniformly on the disk $\{z \mid |z| < R\}$ and defines there a holomorphic function. It follows that $\sum_{n=0}^{\infty} \frac{c_n}{z+n}$ is a meromorphic function on that disk with simple poles at the points of S in $\{z \mid |z| < R\}$. Therefore, $\sum_{n=0}^{\infty} \frac{c_n}{z+n}$ is a meromorphic function with simple poles at the points in S. Therefore, for any $-n \in S$ we have

$$f(z) = \frac{c_n}{z+n} + \sum_{-m \in S-\{n\}} \frac{c_m}{z+m} = \frac{c_n}{z+n} + g(z)$$

where g is holomorphic at $-n$. This implies that $\operatorname{Res}(f, -n) = c_n$. □

Going back to Γ, we have

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt = \int_0^1 t^{z-1} e^{-t}\, dt + \int_1^{\infty} t^{z-1} e^{-t}\, dt.$$

Clearly, the second integral converges for any complex z and represents an entire function. On the other hand, since the exponential function is entire, its Taylor series converges uniformly on compact sets in $\mathbb{C}$, and we have

$$\int_0^1 t^{z-1} e^{-t}\, dt = \int_0^1 t^{z-1} \left(\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} t^p \right) dt$$
$$= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int_0^1 t^{p+z-1}\, dt = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{z+p}$$

for any $z \in \mathbb{C}$. Therefore,

$$\Gamma(z) = \int_1^{\infty} t^{z-1} e^{-t}\, dt + \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{z+p}$$

for any z in the right half-plane. By 1.2.1, the right side of this equation defines a meromorphic function on the complex plane with simple poles at $0, -1, -2, -3, -\cdots$. Hence, we have the following result.

1.2.2. **Theorem.** *The function Γ extends to a meromorphic function on the complex plane. It has simple poles at $0, -1, -2, -3, \cdots$. The residues of Γ are $-p$ are given by*

$$\operatorname{Res}(\Gamma, -p) = \frac{(-1)^p}{p!}$$

for any $p \in \mathbb{Z}_+$.

This result combined with the above calculation immediately implies the following functional equation.

1.2.3. **Proposition.** *For any $z \in \mathbb{C}$ we have*

$$\Gamma(z+1) = z\Gamma(z).$$

1.3. **Another functional equation.** Let $\operatorname{Re} z > 0$. Then

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Let $\operatorname{Re} p > 0$ and $\operatorname{Re} q > 0$. Then, by change of variable $t = u^2$, we get

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt = 2 \int_0^\infty e^{-u^2} u^{2p-1} du.$$

Analogously we have

$$\Gamma(q) = 2 \int_0^\infty e^{-v^2} v^{2q-1} dv.$$

Hence, it follows that

$$\Gamma(p)\Gamma(q) = 4 \int_0^\infty \int_0^\infty e^{-(u^2+v^2)} u^{2p-1} v^{2q-1} du\, dv,$$

and by passing to the polar coordinates by $u = r\cos\varphi$, $v = r \sin\varphi$, we have

$$\begin{aligned} \Gamma(p)\Gamma(q) &= 4 \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(p+q)-1} \cos^{2p-1}\varphi \sin^{2q-1}\varphi \, dr\, d\varphi \\ &= \left(2 \int_0^\infty e^{-r^2} r^{2(p+q)-1}\, dr \right) \cdot \left(2 \int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi \, \sin^{2q-1}\varphi \, d\varphi \right) \\ &= 2\Gamma(p+q) \int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi \, \sin^{2q-1}\varphi \, d\varphi. \end{aligned}$$

We put $s = \sin^2\varphi$ in the integral. Then we have

$$2 \int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi \, \sin^{2q-1}\varphi \, d\varphi = \int_0^1 s^{q-1} (1-s)^{p-1} ds.$$

If we define

$$B(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1} ds$$

for $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, we get the identity

$$B(p,q) = B(q,p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Here B is Euler's *Beta* function.

Let $x \in (0,1)$. Then we have

$$\Gamma(x)\Gamma(1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} = B(x, 1-x) = \int_0^1 s^{x-1}(1-s)^{-x} ds.$$

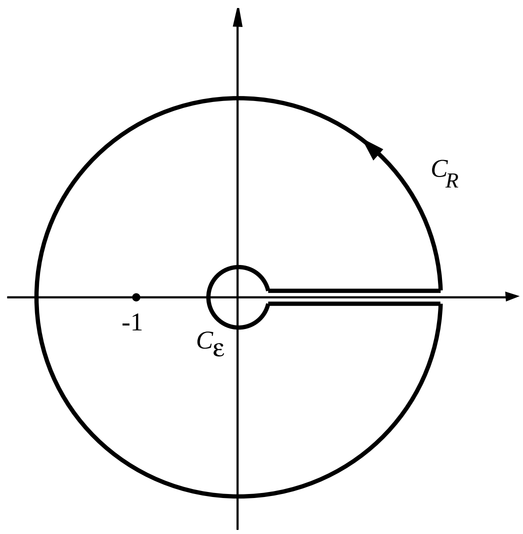
Moreover, if we change the variable $s = \frac{u}{u+1}$, the integral becomes

$$\begin{aligned} \int_0^1 s^{x-1}(1-s)^{-x} ds = \int_0^\infty \frac{u^{x-1}}{(u+1)^{x-1}} \left(1 - \frac{u}{u+1}\right)^{-x} \frac{du}{(u+1)^2} \\ = \int_0^\infty \frac{u^{x-1}}{1+u} du. \end{aligned}$$

1.3.1. **Lemma.** *For $y \in (0,1)$ we have*

$$\int_0^\infty \frac{u^{-y}}{1+u}\, du = \frac{\pi}{\sin \pi y}.$$

Proof. We cut the complex plane along the positive real axis. On this region we define the function $z \longmapsto \frac{z^{-y}}{1+z}$ with the argument of z^{-y} equal to 0 on the upper side of the cut. This function has a first order pole at $z = -1$ with the residue $e^{-\pi i y}$.



If we integrate this function along a path which goes along the upper side of the cut from $\epsilon > 0$ to R, then along the circle C_R of radius R centered at the origin, then along the lower side of the cut from R to ϵ and finally around the origin along the circle C_ϵ of radius ϵ, by the residue theorem we get

$$\int_\epsilon^R \frac{u^{-y}}{1+u}\,du + \int_{C_R} \frac{z^{-y}}{1+z}\,dz - e^{-2\pi i y} \int_\epsilon^R \frac{u^{-y}}{1+u}\,du - \int_{C_\epsilon} \frac{z^{-y}}{1+z}\,dz = 2\pi i e^{-\pi i y}.$$

First we remark that for $z \neq 0$ we have

$$|z^{-y}| = |e^{-y \log z}| = e^{-y\,\mathrm{Re}(\log z)} = e^{-y \log |z|} = |z|^{-y}.$$

Hence, the integrand in the second and fourth integral satisfies

$$\left|\frac{z^{-y}}{1+z}\right| \leq \frac{|z|^{-y}}{|1+z|} \leq \frac{|z|^{-y}}{|1-|z||}.$$

Hence, for small ϵ we have

$$\left|\int_{C_\epsilon} \frac{z^{-y}}{1+z}\,dz\right| \leq 2\pi \frac{\epsilon^{1-y}}{(1-\epsilon)};$$

and for large R we have

$$\left|\int_{C_R} \frac{z^{-y}}{1+z}\,dz\right| \leq 2\pi \frac{R^{1-y}}{(R-1)};$$

Clearly, this implies that $\int_{C_R} \to 0$ as $R \to \infty$ and $\int_{C_\epsilon} \to 0$ as $\epsilon \to 0$. This implies that

$$\left(1 - e^{-2\pi i y}\right) \int_0^\infty \frac{u^{-y}}{1+u}\,du = 2\pi i e^{-\pi i y}.$$

It follows that

$$\left(e^{\pi i y} - e^{-\pi i y}\right) \int_0^\infty \frac{u^{-y}}{1+u}\, du = 2\pi i$$

and finally

$$\int_0^\infty \frac{u^{-y}}{1+u} du = \frac{\pi}{\sin \pi y}.$$

□

This lemma implies that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi(1-x)} = \frac{\pi}{\sin \pi x}.$$

for $x \in (0,1)$. Since both sides are meromorphic, it follows that the following result holds.

1.3.2. **Proposition.** *For all $z \in \mathbb{C}$, we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Since $z \longmapsto \sin \pi z$ is an entire function, the right side obviously has no zeros. So, $\Gamma(z) = 0$ is possible only at points where $z \longmapsto \Gamma(1-z)$ has poles. Since the poles of Γ are at $0, -1, -2, \cdots$, it follows that poles of $z \longmapsto \Gamma(1-z)$ are at $1, 2, 3, \cdots$. At these points $\Gamma(n+1) = n! \neq 0$. Therefore, we proved the following result.

1.3.3. **Theorem.** *The function Γ has no zeros.*

2. Zeta function

2.1. **Meromorphic continuation.** Riemann's zeta function ζ is defined by

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$$

for $\operatorname{Re} z > 1$. In that region the series converges uniformly on compact sets and represents a holomorphic function.

If we consider the expression for the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

for $\operatorname{Re} z > 0$, and change the variable t into $t = ns$ for $n \in \mathbb{N}$, we get

$$\Gamma(z) = n^z \int_0^\infty s^{z-1} e^{-ns} ds,$$

i.e., we have

$$\frac{\Gamma(z)}{n^z} = \int_0^\infty t^{z-1} e^{-nt} dt$$

for any $n \in \mathbb{N}$ and $\operatorname{Re} z > 0$. This implies that, for $\operatorname{Re} z > 1$, we have

$$\begin{aligned}\Gamma(z)\zeta(z) &= \sum_{n=1}^\infty \int_0^\infty t^{z-1} e^{-nt} dt \\ &= \int_0^\infty t^{z-1} \left(\sum_{n=1}^\infty e^{-nt} \right) dt = \int_0^\infty t^{z-1} \frac{e^{-t}}{1-e^{-t}} dt = \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt.\end{aligned}$$

This establishes the following integral representation for the zeta function.

2.1.1. **Lemma.** *For* $\operatorname{Re} z > 1$ *we have*

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt.$$

This integral can be split in to two parts

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \int_0^1 \frac{t^{z-1}}{e^t - 1} dt + \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt.$$

Clearly, the second integral converges for all $z \in \mathbb{C}$ and therefore represents an entire function which we denote by F.

On the other hand, the function $z \longmapsto \frac{1}{e^z - 1}$ has a simple pole at 0 with residue 1. Therefore,

$$\frac{1}{e^z - 1} = \frac{1}{z} + G(z)$$

where G is an meromorphic function with first order poles at $2m\pi i$ with integer $m \neq 0$. It follows that

$$\frac{1}{e^z - 1} = \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n$$

for any $|z| < 2\pi$. By Cauchy's estimates, if we fix $0 < r < 2\pi$, it follows that $|c_n| \leq \frac{M}{r^n}$ for some $M > 0$. In particular, there exists $M > 0$ such that $|c_n| \leq \frac{M}{2^n}$ for all $n \in \mathbb{Z}_+$.

This implies that, for $\operatorname{Re} z > 1$, since the above series converges uniformly on $[0, 1]$, we have

$$\int_0^1 \frac{t^{z-1}}{e^t - 1} dt = \int_0^1 t^{z-2} dt + \int_0^1 \left(\sum_{n=0}^{\infty} c_n t^{z+n-1} \right) dt$$
$$= \frac{1}{z-1} + \sum_{n=0}^{\infty} c_n \int_0^1 t^{z+n-1} dt = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{c_n}{z+n}.$$

Therefore, we have

$$\Gamma(z)\zeta(z) = F(z) + \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{c_n}{z+n}.$$

By 1.2.1, the right side is a meromorphic function, holomorphic for any complex z different form $z = 1, 0, -1, -2, \dots$. Hence, $z \longmapsto \Gamma(z)\zeta(z)$ extends to a meromorphic function in the complex plane with simple pole at 1 and either simple poles or removable singularities at $z = 0, -1, -2, \dots$ depending if $c_n \neq 0$ or $c_n = 0$. Since Γ has no zeros, $z \longmapsto \frac{1}{\Gamma(z)}$ is an entire function. This implies that the zeta function ζ extends to a meromorphic function in the complex plane.

Since $\Gamma(1) = 1$, ζ has a simple pole at 1 with residue equal to 1. Moreover, since Γ has simple poles at $z = 0, -1, -2, \dots$ by 1.2.2, the function $\frac{1}{\Gamma}$ has simple zeros there. It follows that ζ has removable singularities at $z = 0, -1, -2, \dots$. Therefore, we established the following result.

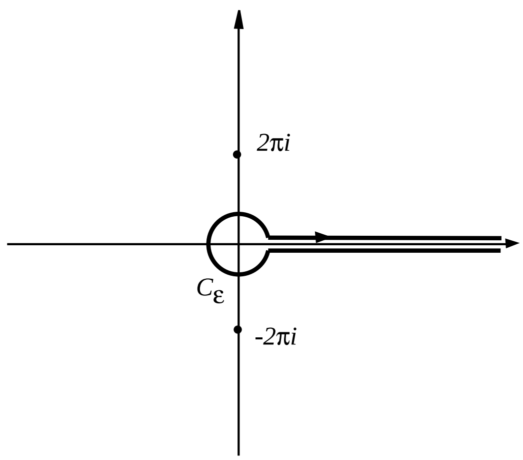
2.1.2. **Theorem.** *The zeta function* ζ *is a meromorphic function with simple pole at* 1*. The residue at this pole is* 1*.*

2.2. **The functional equation.** In this section we prove the following result.

2.2.1. **Theorem.** *For any $z \in \mathbb{C}$, we have*

$$\zeta(z) = \pi^{z-1} 2^z \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z).$$

To establish this functional equation we first establish a variant of 2.1.1.
Let C be a path sketched in the following figure.



We cut the complex plane along positive real axis and consider the integral

$$\int_C \frac{w^{z-1}}{e^w - 1} dw.$$

Here we fix the argument of w to be 0 on the top side of the cut and 2π on the bottom. Clearly, the integrand is holomorphic on the complement of the positive real axes except at zeros of the denominator of the integrand, i.e., at the points $2m\pi i$ for $m \in \mathbb{Z}$.

If the radius ϵ of the arc C_ϵ is less than 2π, it follows immediately from the Cauchy theorem that this integral doesn't depend on ϵ and the distance of the horizontal lines from the positive real axis. Therefore, to evaluate it we can let ϵ and that distance tend to 0.

First we remark that for $w \neq 0$ we have

$$|w^z| = |e^{z \log w}| = e^{\operatorname{Re}(z \log w)} = e^{\operatorname{Re} z \log |w| - \operatorname{Im} z \arg(w)} = |w|^{\operatorname{Re} z} e^{-\operatorname{Im} z \arg(w)}.$$

Since the argument is in $[0, 2\pi]$, for a fixed z, we see that $|w^z| < M|w|^{\operatorname{Re} z}$ for some positive constant M.

We can estimate the integral over C_ϵ as

$$\left| \int_{C_\epsilon} \frac{w^{z-1}}{e^w - 1} dw \right| \leq M \epsilon^{\operatorname{Re} z} \int_0^{2\pi} \frac{1}{|e^{\epsilon e^{i\phi}} - 1|} d\phi.$$

Since $w \longmapsto e^w - 1$ has a simple zero at the origin, $e^w - 1 = w g(w)$ where g is holomorphic near the origin and $g(0) = 1$. It follows that

$$|e^w - 1| \geq \frac{1}{2}|w|$$

for small w. Hence, we have

$$\left| \int_{C_\epsilon} \frac{w^{z-1}}{e^w - 1} dw \right| \leq 4\pi M \epsilon^{\operatorname{Re} z - 1}$$

for small ϵ. In particular, if $\operatorname{Re} z > 1$, we see that the integral over C_ϵ tends to 0 as $\epsilon \to 0$.

Hence, by taking the limit as ϵ goes to zero, and the horizontal lines tend to the real axis, we get

$$\begin{aligned}\int_C \frac{w^{z-1}}{e^w - 1} dw = \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt - e^{2\pi(z-1)i} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt \\ = \left(1 - e^{2\pi i z}\right) \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = -2ie^{i\pi z} \sin(\pi z) \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt \\ = -2ie^{i\pi z} \sin(\pi z) \Gamma(z) \zeta(z)\end{aligned}$$

by 2.1.1. This implies that

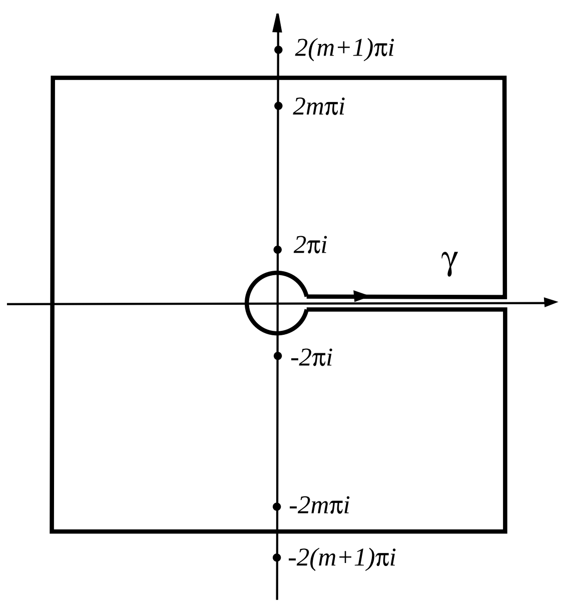
$$-2ie^{i\pi z} \sin(\pi z) \Gamma(z) \zeta(z) = \int_C \frac{w^{z-1}}{e^w - 1} dw$$

for any path C we considered above and $\operatorname{Re} z > 1$. Clearly, the right integral makes sense for arbitrary complex z, hence this equality is an equality of meromorphic functions on $\mathbb{C}$. This yields the following result, which is another integral representation of ζ as a meromorphic function.

2.2.2. **Lemma.** *For any $z \in \mathbb{C}$, we have*

$$\sin(\pi z) \Gamma(z) \zeta(z) = \frac{i}{2} e^{-i\pi z} \int_C \frac{w^{z-1}}{e^w - 1} dw.$$

On the other hand, we can calculate using the residue theorem the integral along the path γ drawn in the following figure. There the outside square passes thru points $(2m+1)\pi i$ and $-(2m+1)\pi i$.



We have

$$\begin{aligned}\int_\gamma \frac{w^{z-1}}{e^w-1}dw &= 2\pi i \sum_{n=1}^m \left((2\pi n)^{z-1}e^{i\frac{\pi(z-1)}{2}} + (2\pi n)^{z-1}e^{3i\frac{\pi(z-1)}{2}}\right)\\ &= 2^z\pi^z i e^{i\pi(z-1)}\left(e^{i\frac{\pi(z-1)}{2}} + e^{-i\frac{\pi(z-1)}{2}}\right)\sum_{n=1}^m n^{z-1}\\ &= -2^{z+1}\pi^z i e^{i\pi z}\cos\left(\frac{\pi(z-1)}{2}\right)\sum_{n=1}^m \frac{1}{n^{1-z}}\\ &= -2^{z+1}\pi^z i e^{i\pi z}\sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^m \frac{1}{n^{1-z}}.\end{aligned}$$

Now we want to estimate the integral along the sides of the square for $\operatorname{Re} z < 0$. For a fixed z, we see as before that there exists $M > 0$, such that on the right vertical side of the square the integrand satisfies the estimate

$$\left|\frac{w^{z-1}}{e^w-1}\right| \le M\frac{|w|^{\operatorname{Re} z-1}}{|e^w-1|}.$$

Moreover, we have

$$|e^w-1| \ge ||e^w|-1| = |e^{\operatorname{Re} w}-1|$$

for any w, hence we have

$$\left|\frac{w^{z-1}}{e^w-1}\right| \le M\frac{|w|^{\operatorname{Re} z-1}}{|e^{\operatorname{Re} w}-1|} \le M\frac{|\operatorname{Re} w|^{\operatorname{Re} z-1}}{|e^{\operatorname{Re} w}-1|}$$

and the right side of the square. This expression clearly tends to zero faster than $\frac{1}{\operatorname{Re} w}$ as $\operatorname{Re} w \to +\infty$. Hence, the integral over the right side tends to zero as the square grows.

On the left side of the square we have the same estimate

$$\left|\frac{w^{z-1}}{e^w-1}\right| \le M\frac{|\operatorname{Re} w|^{\operatorname{Re} z-1}}{|e^{\operatorname{Re} w}-1|}.$$

Since $\operatorname{Re} w$ is negative in this case, for large $|\operatorname{Re} w|$ we have

$$\left|\frac{w^{z-1}}{e^w-1}\right| \le 2M|\operatorname{Re} w|^{\operatorname{Re} z-1}.$$

Since $\operatorname{Re} z < 0$, this bound goes to zero faster that $\frac{1}{|\operatorname{Re} w|}$. On the other hand, the length of the side of the square is $2|\operatorname{Re} w|$. Hence, the integral along the left side also tends to zero as the square grows.

It remains to treat the top and bottom side. As before, we see that on these sides we have

$$\left|\frac{w^{z-1}}{e^w-1}\right| \le M\frac{|w|^{\operatorname{Re} z-1}}{|e^w-1|} \le M\frac{|\operatorname{Im} w|^{\operatorname{Re} z-1}}{|e^w-1|}.$$

On the other hand, the function $w \longmapsto |e^w-1|$ on the horizontal lines tends to ∞ as $\operatorname{Re} w \to +\infty$ and to 1 as $\operatorname{Re} w \to -\infty$. Hence it is bounded from below. Moreover, since it is periodic with period $2\pi i$, shifting the line up and down by $2\pi i$ doesn't change that bound. Hence, in our estimates we can assume that

$$\frac{1}{|e^w-1|} \le C$$

on top and bottom sides of all squares. This implies that

$$\left|\frac{w^{z-1}}{e^w-1}\right| \leq MC|\operatorname{Im} w|^{\operatorname{Re} z-1}$$

on top and bottom sides of all squares. Since $\operatorname{Re} z < 0$ and the length of these sides is $2|\operatorname{Im} w|$, we see that the integrals over these sides tend to zero as the squares grow.

Hence, for $\operatorname{Re} z < 0$, as the squares grow, i.e., as m tends to infinity, the integral over the sides of the square tends to 0 and the integral along γ converges to the integral along the path C from 2.2.2. In addition, the sum on the right side of the above expression converges to the series for $\zeta(1-z)$, i.e., we have

$$-2ie^{\pi iz}\sin(\pi z)\Gamma(z)\zeta(z) = -2^{z+1}\pi^z ie^{i\pi z}\sin\left(\frac{\pi z}{2}\right)\zeta(z-1).$$

Hence, we have

$$\sin(\pi z)\Gamma(z)\zeta(z) = 2^z\pi^z\sin\left(\frac{\pi z}{2}\right)\zeta(1-z).$$

This yields

$$2\sin\left(\frac{\pi z}{2}\right)\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\zeta(z) = 2^z\pi^z\sin\left(\frac{\pi z}{2}\right)\zeta(1-z).$$

and

$$\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\zeta(z) = 2^{z-1}\pi^z\zeta(1-z).$$

for all $z \in \mathbb{C}$. By substituting $1-z$ for z we get

$$2^{-z}\pi^{1-z}\zeta(z) = \cos\left(\frac{\pi(1-z)}{2}\right)\Gamma(1-z)\zeta(1-z)$$

and finally

$$\zeta(z) = 2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z)$$

what completes the proof of the theorem.

2.3. **Euler product formula.** Let P be the set of all prime numbers in $\mathbb{N}$. Assume that $P = \{p_1, p_2, \dots\}$ written in the natural ordering. For any $m \in \mathbb{N}$, let S_m be the subset of $\mathbb{N}$ consisting of all integers which are not divisible by primes $p_1, p_2, \dots, p_m$. Then we claim that, for $\operatorname{Re} z > 1$, we have

$$\zeta(z)\prod_{k=1}^{m}\left(1-\frac{1}{p_k^z}\right) = \sum_{n\in S_m}\frac{1}{n^z}.$$

Clearly, $p_1 = 2$, and we have

$$\left(1-\frac{1}{2^z}\right)\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} - \sum_{n=1}^{\infty}\frac{1}{(2n)^z} = \sum_{n\in S_1}\frac{1}{n^z}.$$

So, the statement holds for $m = 1$.

Assume that the statement holds for m. Then, by the induction assumption, we have

$$\begin{aligned}\left(1-\frac{1}{p_{m+1}^z}\right)\zeta(z)\prod_{k=1}^{m}\left(1-\frac{1}{p_k^z}\right) &= \left(1-\frac{1}{p_{m+1}^z}\right)\sum_{n\in S_m}\frac{1}{n^z}\\ &= \sum_{n\in S_m}\frac{1}{n^z}-\sum_{n\in S_m}\frac{1}{(p_{m+1}n)^z}=\sum_{n\in S_{m+1}}\frac{1}{n^z}.\end{aligned}$$

This establishes the above claim. Therefore, we see that as m tends to ∞ we get the formula

$$\zeta(z)\prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^z}\right)=1,$$

i.e., the following result holds.

2.3.1. **Theorem** (Euler product). *For* $\operatorname{Re} z > 1$ *we have*

$$\zeta(z)=\prod_{p\in P}\frac{1}{\left(1-\frac{1}{p^z}\right)}.$$

The factors in Euler products are nonzero and holomorphic for $\operatorname{Re} z > 0$.

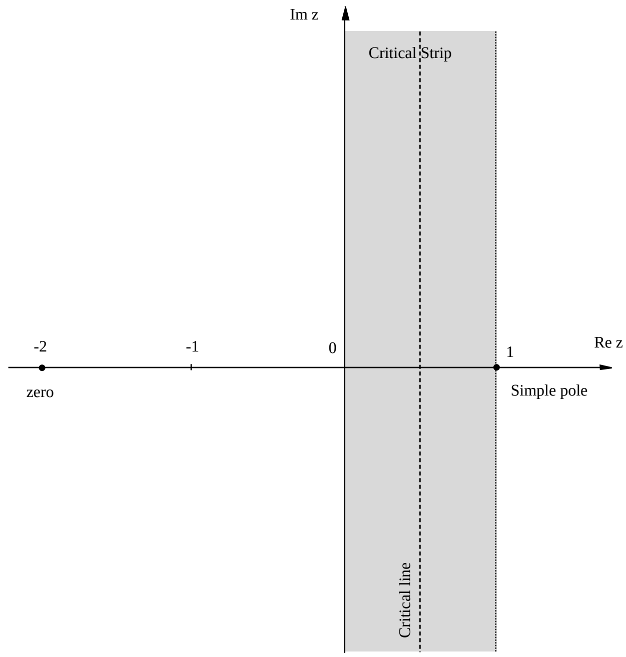
The above observation lead Euler to a proof that the set P is infinite. The finiteness of P would imply that the Euler product is holomorphic for $\operatorname{Re} z > 0$, contradicting the fact that ζ has a pole at 1.[1]

The Euler product formula implies that ζ has no zeros for $\operatorname{Re} z > 1$.

2.4. **Riemann hypothesis.** Since ζ has no zeros for $\operatorname{Re} z > 1$, and Γ has no zeros at all by 1.3.3, we see from the functional equation 2.2.1 that the only zeros of ζ for $\operatorname{Re} z < 0$ come from zeros of the function $z \longmapsto \sin\left(\frac{z\pi}{2}\right)$ at points $-2, -4, \dots$. These are called the *trivial zeros* of ζ.

It follows that all other zeros of ζ have to lie in the strip $0 \le \operatorname{Re} z \le 1$. It is called the *critical strip*.

[1] Actually, Euler didn't use complex variables. One can show that finiteness of primes would contradict the fact that harmonic series diverge.



Riemann conjectured that all zeros of ζ in the critical strip lie on the *critical line* $\operatorname{Re} z = \frac{1}{2}$. This is the *Riemann hypothesis*.

Hadamard and de la Valée-Poussin proved that there are no zeros of ζ on the boundary of the critical strip (i.e., for $\operatorname{Re} z = 0$ and $\operatorname{Re} z = 1$). This implies the Prime Number Theorem conjectured by Gauss.