

NOTES ON REPRESENTATIONS OF COMPACT GROUPS

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1. HAAR MEASURE ON COMPACT GROUPS

1.1. Compact groups. Let G be a group. We say that G is a *topological group* if G is equipped with hausdorff topology such that the multiplication $(g, h) \mapsto gh$ from the product space $G \times G$ into G and the inversion $g \mapsto g^{-1}$ from G into G are continuous functions.

Let G and H be two topological groups. A *morphism of topological groups* $\varphi : G \rightarrow H$ is a group homomorphism which is also continuous.

Topological groups and morphisms of topological groups for the category of topological groups.

Let G be a topological group. Let G^{opp} be the topological space G with the multiplication $(g, h) \mapsto g \star h = h \cdot g$. Then G^{opp} is also a topological group which we call the *opposite group* of G . Clearly, the inverse of an element $g \in G$ is the same as the inverse in G^{opp} . Moreover, the map $g \mapsto g^{-1}$ is an isomorphism of G with G^{opp} . Clearly, we have $(G^{opp})^{opp} = G$.

A topological group G is *compact*, if G is a compact space. The opposite group of a compact group is compact.

We shall need the following fact. Let G be a topological group. We say that a function $\phi : G \rightarrow \mathbb{C}$ is *right (resp. left) uniformly continuous* on G if for any $\epsilon > 0$ there exists an open neighborhood U of 1 such that $|\phi(g) - \phi(h)| < \epsilon$ for any $g, h \in G$ such that $gh^{-1} \in U$ (resp. $g^{-1}h \in U$). Clearly, a right uniformly continuous function on G is left uniformly continuous function on G^{opp} .

1.1.1. Lemma. *Let G be a compact group. Let ϕ be a continuous function on G . Then ϕ is right and left uniformly continuous on G .*

Proof. By the above discussion, it is enough to prove that ϕ is right uniformly continuous.

Let $\epsilon > 0$. Let consider the set $A = \{(g, g') \in G \times G \mid |\phi(g) - \phi(g')| < \epsilon\}$. Then A is an open set in $G \times G$. Let U be an open neighborhood of 1 in G and $B_U = \{(g, g') \in G \times G \mid g'g^{-1} \in U\}$. Since the function $(g, g') \mapsto g'g^{-1}$ is continuous on $G \times G$ the set B_U is open. It is enough to show that there exists an open neighborhood V of 1 in G such that $B_V \subset A$.

Clearly, B_U are open sets containing the diagonal Δ in $G \times G$. Moreover, under the homomorphism κ of $G \times G$ given by $\kappa(g, g') = (g, g'g^{-1})$, $g, g' \in G$, the sets B_U correspond to the sets $G \times U$. In addition, the diagonal Δ corresponds to $G \times \{1\}$. Assume that the open set O corresponds to A .

By the definition of product topology, for any $g \in G$ there exist neighborhoods U_g of 1 and V_g of g such that $V_g \times U_g$ is a neighborhood of $(g, 1)$ contained in O . Clearly, $(V_g; g \in G)$ is an open cover of G . Since G is compact, there exists a finite subcovering $(V_{g_i}; 1 \leq i \leq n)$ of G . Put $U = \bigcap_{i=1}^n U_{g_i}$. Then U is an open

neighborhood of 1 in G . Moreover, $G \times U$ is an open set in $G \times G$ contained in O . Therefore $B_U \subset A$. \square

Therefore, we can say that a continuous function on G is uniformly continuous.

1.2. A compactness criterion. Let X be a compact space. Denote by $C(X)$ the space of all complex valued continuous functions on X . Let $\|f\| = \sup_{x \in X} |f(x)|$ for any $f \in C(X)$. Then $f \mapsto \|f\|$ is a norm on $C(X)$, $C(X)$ is a Banach space.

Let \mathcal{S} be a subset of $C(X)$.

We say that \mathcal{S} is *equicontinuous* if for any $\epsilon > 0$ and $x \in X$ there exists a neighborhood U of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in \mathcal{S}$.

We say that \mathcal{S} is *pointwise bounded* if for any $x \in X$ there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{S}$.

The aim of this section is to establish the following theorem.

1.2.1. Theorem (Arzelà-Ascoli). *Let \mathcal{S} be a pointwise bounded, equicontinuous subset of $C(X)$. Then the closure of \mathcal{S} is a compact subset of $C(X)$.*

Proof. We first prove that \mathcal{S} is bounded in $C(X)$. Let $\epsilon > 0$. Since \mathcal{S} is equicontinuous, for any $x \in X$, there exists an open neighborhood U_x of x such that $y \in U_x$ implies that $|f(y) - f(x)| < \epsilon$ for all $f \in \mathcal{S}$. Since X is compact, there exists a finite set of points $x_1, x_2, \dots, x_n \in X$ such that $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ cover X .

Since \mathcal{S} is pointwise bounded, there exists $M \geq 2\epsilon$ such that $|f(x_i)| \leq \frac{M}{2}$ for all $1 \leq i \leq n$ and all $f \in \mathcal{S}$. Let $x \in X$. Then $x \in U_{x_i}$ for some $1 \leq i \leq n$. Therefore, we have

$$|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < \frac{M}{2} + \epsilon \leq M$$

for all $f \in \mathcal{S}$. It follows that $\|f\| \leq M$ for all $f \in \mathcal{S}$. Hence \mathcal{S} is contained in a closed ball of radius M centered at 0 in $C(X)$.

Now we prove that \mathcal{S} is contained in a finite family of balls of fixed small radius centered in elements of \mathcal{S} . We keep the choices from the first part of the proof. Let $D = \{z \in \mathbb{C} \mid |z| \leq M\}$. Then D is compact. Consider the compact set D^n . It has natural metric given by $d(z, y) = \max_{1 \leq i \leq n} |z_i - y_i|$. There exist points $\alpha_1, \alpha_2, \dots, \alpha_m$ in D^n such that the balls $B_i = \{\beta \in D^n \mid d(\alpha_i, \beta) < \epsilon\}$ cover D^n .

Denote by Φ the map from \mathcal{S} into D^n given by $f \mapsto (f(x_1), f(x_2), \dots, f(x_n))$. Then we can find a subfamily of the above cover of D^n consisting of balls intersecting $\Phi(\mathcal{S})$. After a relabeling, we can assume that these balls are B_i for $1 \leq i \leq k$. Let f_1, f_2, \dots, f_k be functions in \mathcal{S} such that $\Phi(f_i)$ is in the ball B_i for any $1 \leq i \leq k$. Denote by C_i the open ball of radius 2ϵ centered in $\Phi(f_i)$. Let $\beta \in B_i$. Then we have $d(\beta, \alpha_i) < \epsilon$ and $d(\Phi(f_i), \alpha_i) < \epsilon$. Hence, we have $d(\beta, \Phi(f_i)) < 2\epsilon$, i.e., $B_i \subset C_i$. It follows that $\Phi(\mathcal{S})$ is contained in the union of C_1, C_2, \dots, C_k .

Differently put, for any function $f \in \mathcal{S}$, there exists $1 \leq i \leq k$ such that $|f(x_j) - f_i(x_j)| < 2\epsilon$ for all $1 \leq j \leq n$.

Let $x \in X$. Then $x \in U_{x_j}$ for some $1 \leq j \leq n$. Hence, we have

$$|f(x) - f_i(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| < 4\epsilon,$$

i.e., $\|f - f_i\| < 4\epsilon$.

Now we can prove the compactness of the closure $\bar{\mathcal{S}}$ of \mathcal{S} . Assume that $\bar{\mathcal{S}}$ is not compact. Then there exists an open cover \mathcal{U} of $\bar{\mathcal{S}}$ which doesn't contain a finite subcover. By the above remark, $\bar{\mathcal{S}}$ can be covered by finitely many closed balls $\{f \in C(X) \mid \|f - f_i\| \leq 1\}$ with $f_i \in \mathcal{S}$. Therefore, there exists a set K_1

which is the intersection of $\bar{\mathcal{S}}$ with one of the closed balls and which is not covered by a finite subcover of \mathcal{U} . By induction, we can construct a decreasing family $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ of closed subsets of $\bar{\mathcal{S}}$ which are contained in closed balls of radius $\frac{1}{n}$ centered in some point of \mathcal{S} , such that none of K_n is covered by a finite subcover of \mathcal{U} .

Let $(F_n; n \in \mathbb{N})$ be a sequence of functions such that $F_n \in K_n$ for all $n \in \mathbb{N}$. Then $F_p, F_q \in K_n$ for all p, q greater than n . Since K_n are contained in closed balls of radius $\frac{1}{n}$, $\|F_p - F_q\| \leq \frac{2}{n}$ for all p, q greater than n . Hence, (F_n) is a Cauchy sequence in $C(X)$. Therefore, it converges to a function $F \in C(X)$. This function is in $\bar{\mathcal{S}}$ and therefore in one element V of the open cover \mathcal{U} . Therefore, for sufficiently large n , there exists a closed ball of radius $\frac{2}{n}$ centered in F which is contained in V . Since F is also in K_n , we see that K_n is in V . This clearly contradicts our construction of K_n . It follows that $\bar{\mathcal{S}}$ must be compact. \square

1.3. Haar measure on compact groups. Let $\mathcal{C}_{\mathbb{R}}(G)$ be the space of real valued functions on G . For any function $f \in \mathcal{C}_{\mathbb{R}}(G)$ we define the *maximum* $M(f) = \max_{g \in G} f(g)$ and *minimum* $m(f) = \min_{g \in G} f(g)$. Moreover, we denote by $V(f) = M(f) - m(f)$ the *variation* of f .

Clearly, the function f is constant on G if and only if $V(f) = 0$.

Let $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$ be two functions such that $\|f - f'\| < \epsilon$. Then

$$f(g) - \epsilon < f'(g) < f(g) + \epsilon$$

for all $g \in G$. This implies that

$$m(f) - \epsilon < f'(g) < M(f) + \epsilon$$

for all $g \in G$, and

$$m(f) - \epsilon < m(f') < M(f') < M(f) + \epsilon.$$

Hence

$$V(f') = M(f') - m(f') < M(f) - m(f) + 2\epsilon = V(f) + 2\epsilon,$$

i.e., $V(f') - V(f) < 2\epsilon$. By symmetry, we also have $V(f) - V(f') < 2\epsilon$. It follows that $|V(f) - V(f')| < 2\epsilon$.

Therefore, we have the following result.

1.3.1. Lemma. *The variation V is a continuous function on $\mathcal{C}_{\mathbb{R}}(G)$.*

Let $f \in \mathcal{C}_{\mathbb{R}}(G)$ and $\mathbf{a} = (a_1, a_2, \dots, a_n)$ a finite sequence of points in G . We define the (*right*) *mean value* $\mu(f, \mathbf{a})$ of f with respect to \mathbf{a} as

$$\mu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^n f(ga_i)$$

for all $g \in G$. Clearly, $\mu(f, \mathbf{a})$ is a continuous real function on G .

If f is a constant function, $\mu(f, \mathbf{a}) = f$.

Clearly, mean value $f \mapsto \mu(f, \mathbf{a})$ is a linear map. Moreover, we have the following result.

1.3.2. Lemma. (i) *The linear map $f \mapsto \mu(f, \mathbf{a})$ is continuous. More precisely, we have*

$$\|\mu(f, \mathbf{a})\| \leq \|f\|$$

for any $f \in \mathcal{C}_{\mathbb{R}}(G)$;

- (ii) $M(\mu(f, \mathbf{a})) \leq M(f)$
for any $f \in C_{\mathbb{R}}(G)$;
- (iii) $m(\mu(f, \mathbf{a})) \geq m(f)$
for any $f \in C_{\mathbb{R}}(G)$;
- (iv) $V(\mu(f, \mathbf{a})) \leq V(f)$
for any $f \in C_{\mathbb{R}}(G)$.

Proof. (i) Clearly, we have

$$\|\mu(f, \mathbf{a})\| = \max_{g \in G} |\mu(f, \mathbf{a})| \leq \frac{1}{n} \sum_{g \in G} \max_{g \in G} |f(ga_i)| = \|f\|.$$

(ii) We have

$$M(\mu(f, \mathbf{a})) = \frac{1}{n} \max_{g \in G} \left(\sum_{i=1}^n f(ga_i) \right) \leq \frac{1}{n} \sum_{i=1}^n \max_{g \in G} f(ga_i) = M(f).$$

(iii) We have

$$m(\mu(f, \mathbf{a})) = \frac{1}{n} \min_{g \in G} \left(\sum_{i=1}^n f(ga_i) \right) \geq \frac{1}{n} \sum_{i=1}^n \min_{g \in G} f(ga_i) = m(f).$$

(iv) By (ii) and (iii), we have

$$V(\mu(f, \mathbf{a})) = M(\mu(f, \mathbf{a})) - m(\mu(f, \mathbf{a})) \leq M(f) - m(f) = V(f).$$

□

Denote by \mathcal{M}_f the set of mean values of f for all finite sequences in G .

1.3.3. Lemma. *The set of functions \mathcal{M}_f is uniformly bounded and equicontinuous.*

Proof. By 1.3.2 (ii) and (iii), it follows that

$$m(f) \leq m(\mu(f, \mathbf{a})) \leq \mu(f, \mathbf{a})(g) \leq M(\mu(f, \mathbf{a})) \leq M(f).$$

This implies that \mathcal{M}_f is uniformly bounded.

Now we want to prove that \mathcal{M}_f is equicontinuous. First, by 1.1.1, the function f is uniformly continuous. Hence, for any $\epsilon > 0$, there exists an open neighborhood U of 1 in G such that $|f(g) - f(h)| < \epsilon$ if $gh^{-1} \in U$. Since, this implies that $(ga_i)(ha_i)^{-1} = gh^{-1} \in U$ for any $1 \leq i \leq n$, we see that

$$|\mu(f, \mathbf{a})(g) - \mu(f, \mathbf{a})(h)| = \frac{1}{n} \left| \sum_{i=1}^n (f(ga_i) - f(ha_i)) \right| \leq \frac{1}{n} \sum_{i=1}^n |f(ga_i) - f(ha_i)| < \epsilon$$

for $g \in hU$. Hence, the family \mathcal{M}_f is equicontinuous. □

By 1.2.1, we have the following consequence.

1.3.4. Lemma. *The set \mathcal{M}_f of all right mean values of f has compact closure in $C_{\mathbb{R}}(G)$.*

We need another result on variation of mean value functions. Clearly, if f is a constant function $\mu(f, \mathbf{a}) = f$ for any \mathbf{a} .

1.3.5. Lemma. *Let f be a function in $\mathcal{C}_{\mathbb{R}}(G)$. Assume that f is not a constant. Then there exists \mathbf{a} such that $V(\mu(f, \mathbf{a})) < V(f)$.*

Proof. Since f is not constant, we have $m(f) < M(f)$. Let C be such that $m(f) < C < M(f)$. Then there exists an open set V in G such that $f(g) \leq C$ for all $g \in V$. Since the right translates of V cover G , by compactness of G we can find $\mathbf{a} = (a_1, a_2, \dots, a_n)$ such that $(Va_i^{-1}, 1 \leq i \leq n)$ is an open cover of G . For any $g \in Va_i^{-1}$ we have $ga_i \in V$ and $f(ga_i) \leq C$. Hence, we have

$$\begin{aligned} \mu(f, \mathbf{a})(g) &= \frac{1}{n} \sum_{j=1}^n f(ga_j) = \frac{1}{n} \left(f(ga_i) + \sum_{j \neq i} f(ga_j) \right) \\ &\leq \frac{1}{n} (C + (n-1)M(f)) < M(f). \end{aligned}$$

On the other hand, by 1.3.2.(iii) we know that $m(\mu(f, \mathbf{a})) \geq m(f)$ for any \mathbf{a} . Hence we have

$$V(\mu(f, \mathbf{a})) = M(\mu(f, \mathbf{a})) - m(\mu(f, \mathbf{a})) < M(f) - m(f) = V(f). \quad \square$$

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_m)$ be two finite sequences in G . We define $\mathbf{a} \cdot \mathbf{b} = (a_i b_j; 1 \leq i \leq n, 1 \leq j \leq m)$.

1.3.6. Lemma. *We have*

$$\mu(\mu(f, \mathbf{b}), \mathbf{a}) = \mu(f, \mathbf{b} \cdot \mathbf{a}).$$

Proof. We have

$$\mu(\mu(f, \mathbf{b}), \mathbf{a}) = \frac{1}{m} \sum_{i=1}^m \mu(f, \mathbf{b})(ga_i) = \frac{1}{nm} \sum_{j=1}^m \sum_{i=1}^n f(ga_i b_j) = \mu(f, \mathbf{a} \cdot \mathbf{b}). \quad \square$$

1.3.7. Lemma. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\overline{\mathcal{M}_f}$ contains a constant function on G .*

Proof. By 1.3.4, we know that $\overline{\mathcal{M}_f}$ is compact. Since, by 1.3.1, the variation V is continuous on $\mathcal{C}_{\mathbb{R}}(G)$, it attains its minimum α at some $\varphi \in \overline{\mathcal{M}_f}$.

Assume that φ is not a constant. By 1.3.5, there exists \mathbf{a} such that $V(\mu(\varphi, \mathbf{a})) < V(\varphi)$. Let $\alpha - V(\mu(\varphi, \mathbf{a})) = \epsilon > 0$.

Since V and $\mu(\cdot, \mathbf{a})$ are continuous maps by 1.3.1 and 1.3.2.(i), this implies that there is \mathbf{b} such that $|V(\mu(\varphi, \mathbf{a})) - V(\mu(\mu(f, \mathbf{b}), \mathbf{a}))| < \frac{\epsilon}{2}$. Therefore, we have

$$V(\mu(\mu(f, \mathbf{b}), \mathbf{a})) \leq V(\mu(\varphi, \mathbf{a})) + \frac{\epsilon}{2} = \alpha - \frac{\epsilon}{2}.$$

By 1.3.6, we have

$$V(\mu(f, \mathbf{a} \cdot \mathbf{b})) < \alpha - \frac{\epsilon}{2}$$

contrary to our choice of α .

It follows that φ is a constant function. In addition $\alpha = 0$. □

Consider now left mean values of a function $f \in \mathcal{C}_{\mathbb{R}}(G)$. We define the left mean value of f with respect to $\mathbf{a} = (a_1, a_2, \dots, a_n)$ as the function

$$\nu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^n f(a_i g)$$

for $g \in G$. We denote by \mathcal{N}_f the set of all left mean values of f .

Let G^{opp} be the compact group opposite to G . Then the left mean values of f on G are the right mean values of f on G^{opp} .

Hence, from 1.3.7, we deduce the following result.

1.3.8. Lemma. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure \mathcal{N}_f contains a constant function on G .*

By direct calculation we get the following result.

1.3.9. Lemma. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$ we have*

$$\nu(\mu(f, \mathbf{a}), \mathbf{b}) = \mu(\nu(f, \mathbf{b}), \mathbf{a})$$

for any two finite sequences \mathbf{a} and \mathbf{b} in G .

Proof. We have

$$\begin{aligned} \nu(\mu(f, \mathbf{a}), \mathbf{b})(g) &= \frac{1}{m} \sum_{j=1}^m \mu(f, \mathbf{a})(b_j g) = \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m f(b_j g a_i) \\ &= \frac{1}{n} \sum_{i=1}^n \nu(f, \mathbf{b})(g a_i) = \mu(\nu(f, \mathbf{b}), \mathbf{a})(g) \end{aligned}$$

for any $g \in G$. □

Putting together these results, we finally get the following.

1.3.10. Proposition. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure \mathcal{M}_f contains a unique function constant on G .*

This function is also the unique constant function in \mathcal{N}_f .

Proof. Let φ and ψ be two constant functions such that φ is in the closure of \mathcal{M}_f and ψ is in the closure of \mathcal{N}_f . For any $\epsilon > 0$ we have \mathbf{a} and \mathbf{b} such that $\|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}$ and $\|\nu(f, \mathbf{b}) - \psi\| < \frac{\epsilon}{2}$.

On the other hand, we have

$$\begin{aligned} \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| &= \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \nu(\varphi, \mathbf{b})\| \\ &= \|\nu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| \leq \|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}. \end{aligned}$$

In the same way, we also have

$$\begin{aligned} \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \psi\| &= \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \mu(\psi, \mathbf{a})\| \\ &= \|\mu(\nu(f, \mathbf{b}) - \psi, \mathbf{a})\| \leq \|\nu(f, \mathbf{b}) - \psi\| < \frac{\epsilon}{2}. \end{aligned}$$

By 1.3.9, this immediately yields

$$\|\varphi - \psi\| \leq \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| + \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \psi\| < \epsilon.$$

This implies that $\varphi = \psi$. Therefore, any constant function in the closure of \mathcal{M}_f has to be equal to ψ . □

The value of the unique constant function in the closure of \mathcal{M}_f is denoted by $\mu(f)$ and called *the mean value of f on G* . In this way, we get a function $f \mapsto \mu(f)$ on $\mathcal{C}_{\mathbb{R}}(G)$.

1.3.11. Lemma. *The function μ is a continuous linear form on $\mathcal{C}_{\mathbb{R}}(G)$.*

To prove this result we need some preparation.

1.3.12. Lemma. *Let $f \in \mathcal{C}_{\mathbb{R}}(G)$. Then, for any \mathbf{a} we have*

$$\mu(\mu(f, \mathbf{a})) = \mu(f).$$

Proof. Let $\mu(f) = \alpha$. Let φ be the function equal to α everywhere on G . Fix $\epsilon > 0$. Then there exists a finite sequence \mathbf{b} such that

$$\|\nu(f, \mathbf{b}) - \varphi\| < \epsilon.$$

This implies that

$$\|\nu(f - \varphi, \mathbf{b})\| = \|\nu(f, \mathbf{b}) - \nu(\varphi, \mathbf{b})\| = \|\nu(f, \mathbf{b}) - \varphi\| < \epsilon.$$

This, by 1.3.2.(i), implies that

$$\|\mu(\nu(f - \varphi, \mathbf{b}), \mathbf{a})\| \leq \|\nu(f - \varphi, \mathbf{b})\| < \epsilon$$

for any finite sequence \mathbf{a} .

By 1.3.9, we have

$$\|\nu(\mu(f - \varphi, \mathbf{a}), \mathbf{b})\| = \|\mu(\nu(f - \varphi, \mathbf{b}), \mathbf{a})\| < \epsilon,$$

and

$$\|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| = \|\nu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| = \|\nu(\mu(f - \varphi, \mathbf{a}), \mathbf{b})\| < \epsilon.$$

Therefore, if we fix \mathbf{a} , we see that φ is in the closure of $\mathcal{N}_{\mu(f, \mathbf{a})}$. By 1.3.10, this proves our assertion. \square

Let f and f' be two functions in $\mathcal{C}_{\mathbb{R}}(G)$. Let $\alpha = \mu(f)$ and $\beta = \mu(f')$. Denote by φ and ψ the corresponding constant functions. Let $\epsilon > 0$.

Clearly, there exists \mathbf{b} such that

$$\|\mu(f, \mathbf{b}) - \varphi\| < \frac{\epsilon}{2}.$$

This, by 1.3.2.(ii) implies, that we have

$$\|\mu(\mu(f, \mathbf{b}), \mathbf{a}) - \varphi\| = \|\mu(\mu(f, \mathbf{b}) - \varphi, \mathbf{a})\| < \frac{\epsilon}{2}.$$

for arbitrary \mathbf{a} . By 1.3.6, this in turn implies that

$$\|\mu(f, \mathbf{a} \cdot \mathbf{b}) - \varphi\| < \frac{\epsilon}{2}.$$

On the other hand, by 1.3.12, we have $\mu(\mu(f', \mathbf{b})) = \mu(f') = \beta$. Therefore, there exists a finite sequence \mathbf{a} such that

$$\|\mu(\mu(f', \mathbf{b}), \mathbf{a}) - \psi\| < \frac{\epsilon}{2}.$$

This, by 1.3.6, implies that

$$\|\mu(f', \mathbf{a} \cdot \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.$$

Hence, we have

$$\|\mu(f + f', \mathbf{a} \cdot \mathbf{b}) - (\varphi + \psi)\| \leq \|\mu(f, \mathbf{a} \cdot \mathbf{b}) - \varphi\| + \|\mu(f', \mathbf{a} \cdot \mathbf{b}) - \psi\| < \epsilon.$$

Therefore, $\varphi + \psi$ is in the closure of $\mathcal{M}_{f+f'}$. Hence we have

$$\mu(f + f') = \alpha + \beta = \mu(f) + \mu(f').$$

It follows that μ is additive.

Let $c \in \mathbb{R}$ and $f \in C_{\mathbb{R}}(G)$. Then $\mu(cf, \mathbf{a}) = c\mu(f, \mathbf{a})$ for any \mathbf{a} . Therefore, $\mathcal{M}_{cf} = c\mathcal{M}_f$. This immediately implies that $\mu(cf) = c\mu(f)$. Therefore μ is a linear form.

On the other hand, by 1.3.2(ii), we have

$$\mu(f, \mathbf{a})(g) \leq M(f)$$

for any \mathbf{a} and $g \in G$. This in turn implies that $\mu(f) \leq M(f) \leq \|f\|$. On the other hand, this implies that

$$-\mu(f) = \mu(-f) \leq \|-f\| = \|f\|.$$

Hence, we have $|\mu(f)| \leq \|f\|$ for any $f \in C_{\mathbb{R}}(G)$, i.e. μ is a continuous linear form.

This completes the proof of 1.3.11.

Assume that f is a function in $C_{\mathbb{R}}(G)$ such that $f(g) \geq 0$ for all $g \in G$. Then $\mu(f, \mathbf{a})(g) \geq 0$ for any \mathbf{a} and $g \in G$. Hence, any function $\phi \in \mathcal{M}_f$ satisfies $\phi(g) \geq 0$ for all $g \in G$. This immediately implies that $\phi(g) \geq 0$, $g \in G$, for any ϕ in the closure of \mathcal{M}_f . It follows that $\mu(f) \geq 0$. Hence, we say that μ is a *positive* linear form.

By Riesz representation theorem, the linear form $\mu : C_{\mathbb{R}}(G) \rightarrow \mathbb{R}$ defines a regular positive measure μ on G such that

$$\mu(f) = \int_G f d\mu.$$

Clearly, we have

$$\mu(G) = \int_G d\mu = \mu(1) = 1.$$

so we say that μ is *normalized*.

Denote by R (resp. L) the *right regular representation* (resp. *left regular representation*) of G on $C(G)$ given by $(R(g)f)(h) = f(hg)$ (resp. $(L(g)f)(h) = f(g^{-1}h)$) for any $f \in C(G)$ and $g, h \in G$.

1.3.13. Lemma. *Let $f \in C_{\mathbb{R}}(G)$ and $g \in G$. Then*

$$\mu(R(g)f) = \mu(L(g)f) = \mu(f).$$

Proof. Let $\mathbf{g} = (g)$. Clearly, we have

$$\mu(f, \mathbf{g})(h) = f(hg) = (R(g)f)(h)$$

for all $h \in G$, i.e., $R(g)f = \mu(f, \mathbf{g})$. By 1.3.12, we have

$$\mu(R(g)f) = \mu(\mu(f, \mathbf{g})) = \mu(f).$$

This statement for G^{opp} implies the other equality. \square

We say that the linear form μ is *biinvariant*, i.e., *right invariant* and *left invariant*.

This in turn implies that the measure μ is *biinvariant*, i.e., we have the following result.

1.3.14. **Lemma.** *Let A be a measurable set in G . Then gA and Ag are also measurable and*

$$\mu(gA) = \mu(Ag) = \mu(A)$$

for any $g \in G$.

Proof. Since $C_{\mathbb{R}}(G)$ is dense in $L^1(\mu)$, the invariance from 1.3.13 holds for any function $f \in L^1(\mu)$. Applying it to the characteristic function of the set A implies the result. \square

A normalized biinvariant positive measure μ on G is called a *Haar measure* on G .

We proved the existence part of the following result.

1.3.15. **Theorem.** *Let G be a compact group. Then there exists a unique Haar measure μ on G .*

Proof. We constructed a Haar measure on G .

It remains to prove the uniqueness. Let ν be another Haar measure on G . Then, by left invariance, we have

$$\int_G \mu(f, \mathbf{a}) d\nu = \frac{1}{n} \sum_{i=1}^n \int_G f(ga_i) d\nu(g) = \int_G f d\nu$$

for any \mathbf{a} . Hence the integral with respect to ν is constant on \mathcal{M}_f . By continuity, it is also constant on its closure. Therefore, we have

$$\int_G f d\nu = \mu(f) \int_G d\nu = \mu(f) = \int_G f d\mu$$

for any $C_{\mathbb{R}}(G)$. This in turn implies that $\nu = \mu$. \square

1.3.16. **Lemma.** *Let μ be the Haar measure on G . Let U be a nonempty open set in G . Then $\mu(U) > 0$.*

Proof. Since U is nonempty, $(Ug; g \in G)$ is an open cover of G . It contains a finite subcover $(Ug_i; 1 \leq i \leq n)$. Therefore we have

$$1 = \mu(G) = \mu\left(\bigcup_{i=1}^n Ug_i\right) \leq \sum_{i=1}^n \mu(Ug_i) = \sum_{i=1}^n \mu(U) = n\mu(U)$$

by 1.3.14. This implies that $\mu(U) \geq \frac{1}{n}$. \square

1.3.17. **Lemma.** *Let f be a continuous function on G . Then*

$$\int_G f(g^{-1}) d\mu(g) = \int_G f(g) d\mu(g).$$

Proof. Clearly, it is enough to prove the statement for real-valued functions. Therefore, we can consider the linear form $\nu : f \mapsto \int_G f(g^{-1}) d\mu(g)$. Clearly, this a positive continuous linear form on $C_{\mathbb{R}}(G)$. Moreover,

$$\begin{aligned} \nu(f) &= \int_G f(h^{-1}) d\mu(h) = \int_G f((hg)^{-1}) d\mu(h) \\ &= \int_G f(g^{-1}h^{-1}) d\mu(h) = \int_G (L(g)f)(h^{-1}) d\mu(h) = \nu(L(g)f) \end{aligned}$$

and

$$\begin{aligned} \nu(f) &= \int_G f(h^{-1}) d\mu(h) = \int_G f((g^{-1}h)^{-1}) d\mu(h) \\ &= \int_G f(h^{-1}g) d\mu(h) = \int_G (R(g)f)(h^{-1}) d\mu(h) = \nu(R(g)f) \end{aligned}$$

for any $g \in G$. Hence, this linear form is left and right invariant. By the uniqueness of the Haar measure we get the statement. \square

2. ALGEBRA OF MATRIX COEFFICIENTS

2.1. Topological vector spaces. Let E be a vector space over \mathbb{C} . We say that E is a *topological vector space* over \mathbb{C} , if it is also equipped with a topology such that the functions $(u, v) \mapsto u + v$ from $E \times E$ into E , and $(\alpha, u) \mapsto \alpha u$ from $\mathbb{C} \times E$ into E are continuous.

A morphism $\varphi : E \rightarrow F$ of topological vector space E into F is a continuous linear map from E to F .

We say that E is a hausdorff topological vector space if the topology of E is hausdorff.

The vector space \mathbb{C}^n with its natural topology is a hausdorff topological vector space.

Let E be a topological vector space and F a vector subspace of E . Then F is a topological vector space with the induced topology. Moreover, if E is hausdorff, F is also hausdorff.

2.1.1. Lemma. *Let E be a topological vector space over \mathbb{C} . Then the following conditions are equivalent:*

- (i) E is hausdorff;
- (ii) $\{0\}$ is a closed set in E .

Proof. Assume that E is hausdorff. Let $v \in E$, $v \neq 0$. Then there exist open neighborhoods U of 0 and V of v such that $U \cap V = \emptyset$. In particular, $V \subset E - \{0\}$. Hence, $E - \{0\}$ is an open set. This implies that $\{0\}$ is closed.

Assume now that $\{0\}$ is closed in E . Then $E - \{0\}$ is an open set. Let u and v be different vectors in E . Then $u - v \neq 0$. Since the function $(x, y) \mapsto x + y$ is continuous, there exist open neighborhoods U of u and V of v such that $U - V \subset E - \{0\}$. This in turn implies that $U \cap V = \emptyset$. \square

The main result of this section is the following theorem. It states that hausdorff finite-dimensional topological vector spaces have unique topology.

2.1.2. Theorem. *Let E be a finite-dimensional hausdorff topological vector space over \mathbb{C} . Let v_1, v_2, \dots, v_n be a basis of E . Then the linear map $\mathbb{C}^n \rightarrow E$ given by*

$$(c_1, c_2, \dots, c_n) \mapsto \sum_{i=1}^n c_i v_i$$

is an isomorphism of topological vector spaces.

Clearly, the map

$$\phi(z) = \sum_{i=1}^n z_i v_i,$$

for all $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, is a continuous linear isomorphism of \mathbb{C}^n onto E . Therefore, it is enough to show that that map is also open.

We start with an elementary lemma.

2.1.3. Lemma. *Let E and F be topological vector spaces and $\phi : E \rightarrow F$ a linear map. Let $(U_i, i \in I)$ be a fundamental system of neighborhoods of 0 in E . If $\phi(U_i)$, $i \in I$, are neighborhoods of 0 in F , ϕ is an open map.*

Proof. Let U be an open set in E . For any $u \in U$ there exists $i \in I$ such that $u + U_i$ is a neighborhood of u contained in U . Therefore, $\phi(u + U_i) = \phi(u) + \phi(U_i)$ is a neighborhood of $\phi(u)$ contained in $\phi(U)$. Hence, $\phi(u)$ is an interior point of $\phi(U)$. This implies that $\phi(U)$ is open. \square

We consider on \mathbb{C}^n the standard euclidean norm $\|\cdot\|$. Let $B_1 = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the open unit ball in \mathbb{C}^n . Then the balls $\{\epsilon B_1 \mid 0 < \epsilon < \infty\}$ form a fundamental system of neighborhoods of 0 in \mathbb{C}^n . By 2.1.3, to show that the above map $\phi : \mathbb{C}^n \rightarrow E$ is open it is enough to show that $\phi(B_1)$ is a neighborhood of 0 in E .

Let $S = \{z \in \mathbb{C}^n \mid \|z\| = 1\}$ be the unit sphere in \mathbb{C}^n . Then, S is a bounded and closed set in \mathbb{C}^n . Hence it is compact. This implies that $\phi(S)$ is a compact set in E . Since 0 is not in S , 0 is not in $\phi(S)$. Since E is hausdorff, $\phi(S)$ is closed and $E - \phi(S)$ is an open neighborhood of 0 in E . By continuity of multiplication by scalars at $(0, 0)$, there exists $\epsilon > 0$ and an open neighborhood U of 0 in E such that $zU \subset E - \phi(S)$, i.e., $zU \cap \phi(S) = \emptyset$ for all $|z| \leq \epsilon$.

Let $v \in U - \{0\}$. Then we have

$$v = \sum_{i=1}^n c_i v_i.$$

Let $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$. Then, $\frac{1}{\|c\|}c \in S$, and $\frac{1}{\|c\|}v \in \phi(S)$. By our construction, we must have $\frac{1}{\|c\|} > \epsilon$. Hence, we have $\|c\| < \frac{1}{\epsilon}$, i.e., $c \in B_{\frac{1}{\epsilon}}$. This in turn yields $v \in \phi(\frac{1}{\epsilon}B_1)$. Therefore, we have

$$\phi\left(\frac{1}{\epsilon}B_1\right) \supset U,$$

i.e., $\phi(B_1) \supset \epsilon U$. Hence, $\phi(B_1)$ is a neighborhood of 0 in E . This completes the proof of 2.1.2.

2.1.4. Corollary. *Let E be a hausdorff topological vector space over \mathbb{C} . Let F be a finite-dimensional vector subspace of E . Then F is closed in E .*

Proof. Clearly, the topology of E induces a structure of hausdorff topological vector space on F . Let v_1, v_2, \dots, v_n be a basis of F . Assume that F is not closed. Let w be a vector in the closure of F which is not in F . Then w is linearly independent of v_1, v_2, \dots, v_n . Let F' be the direct sum of F and $\mathbb{C}w$. Then F' is a $(n+1)$ -dimensional hausdorff topological vector space. By 2.1.2, we know that

$$(c_1, c_2, \dots, c_n, c_{n+1}) \mapsto \sum_{i=1}^n c_i v_i + c_{n+1} w$$

is an isomorphism of the topological vector space \mathbb{C}^{n+1} onto F' . This isomorphism maps $\mathbb{C}^n \times \{0\}$ onto F . Therefore, F is closed in F' , and w is not in the closure of F . Hence, we have a contradiction. \square

2.1.5. Lemma. *Let E and F be two finite-dimensional hausdorff topological vector spaces. Then any linear map $\phi : E \rightarrow F$ is continuous.*

2.2. Representations on topological vector spaces. Let G be a compact group. Let E be a hausdorff topological vector space over \mathbb{C} . We denote by $\text{GL}(E)$ the group of all automorphisms of E .

A (continuous) representation of G on E is a group homomorphism $\pi : G \rightarrow \text{GL}(E)$ such that $(g, v) \mapsto \pi(g)v$ is continuous from $G \times E$ into E .

2.2.1. Lemma. *Let E be a Banach space and $\pi : G \rightarrow \text{GL}(E)$ a homomorphism such that $g \mapsto \pi(g)v$ is continuous function from G into E for all $v \in E$. Then (π, E) is a representation of G on E .*

Proof. Assume that the function $g \mapsto \pi(g)v$ is continuous for any $v \in V$. Then the function $g \mapsto \|\pi(g)v\|$ is continuous on G . Since G is compact, there exists M such that $\|\pi(g)v\| < M$ for all $g \in G$. By Banach-Steinhaus theorem, we see that the function $g \mapsto \|\pi(g)\|$ is bounded on G .

Pick $C > 0$ such that $\|\pi(g)\| \leq C$ for all $g \in G$. Then we have

$$\begin{aligned} \|\pi(g)v - \pi(g')v'\| &= \|(\pi(g)v - \pi(g')v) + \pi(g')(v - v')\| \\ &\leq \|\pi(g)v - \pi(g')v\| + \|\pi(g')\|\|v - v'\| \leq \|\pi(g)v - \pi(g')v\| + C\|v - v'\| \end{aligned}$$

for all $g, g' \in G$ and $v, v' \in E$. This clearly implies the continuity of the function $(g, v) \mapsto \pi(g)v$. \square

If E is a finite-dimensional hausdorff topological vector space, by 2.1.5, any linear automorphism of E is automatically an automorphism of topological vector spaces. Therefore $\text{GL}(E)$ is just the group of all linear automorphisms of E as before.

Moreover, since the topology of E is described by the euclidean norm and E is a Banach space with respect to it, by 2.2.1, the only additional condition for a representation of G is the continuity of the function $g \mapsto \pi(g)v$ for any $v \in E$. This implies the following result.

2.2.2. Lemma. *Let E be a finite-dimensional hausdorff topological vector space and π a homomorphism of G into $\text{GL}(E)$. Let v_1, v_2, \dots, v_n be a basis of E .*

- (i) (π, E) is a representation of G on E ;
- (ii) all matrix coefficients of $\pi(g)$ with respect to the basis v_1, v_2, \dots, v_n are continuous functions on G .

2.3. Algebra of matrix coefficients. Let G be a compact group. The Banach space $C(G)$ is an commutative algebra with pointwise multiplication of functions, i.e., $(\psi, \phi) \mapsto \psi \cdot \phi$ where $(\psi \cdot \phi)(g) = \psi(g)\phi(g)$ for any $g \in G$.

First, we remark the following fact.

2.3.1. Lemma. *R and L are representations of G on $C(G)$.*

Proof. Clearly, we have

$$\|R(g)\phi\| = \max_{h \in G} |(R(g)\phi)(h)| = \max_{h \in G} |\phi(hg)| = \max_{h \in G} |\phi(h)| = \|\phi\|.$$

Hence $R(g)$ is a continuous linear map on $C(G)$. Its inverse is $R(g^{-1})$, so $R(g) \in \text{GL}(C(G))$.

By 2.2.1, it is enough to show that the function $g \mapsto R(g)\phi$ is continuous for any function $\phi \in C(G)$.

By 1.1.1, ϕ is uniformly continuous, i.e., there exists a neighborhood U of 1 in G such that $g^{-1}g' \in U$ implies $|\phi(hg) - \phi(hg')| < \epsilon$ for all $h \in G$. Hence, we have

$$\|R(g)\phi - R(g')\phi\| = \max_{h \in G} |(R(g)\phi)(h) - (R(g')\phi)(h)| = \max_{h \in G} |\phi(hg) - \phi(hg')| < \epsilon$$

for $g' \in gU$. Hence, the function $g \rightarrow R(g)\phi$ is continuous.

The proof for L is analogous. \square

We say that the function $\phi \in C(G)$ is right (resp. left) G -finite if the vectors $\{R(g)\phi; g \in G\}$ (resp. $\{L(g)\phi; g \in G\}$) span a finite-dimensional subspace of $C(G)$.

2.3.2. Lemma. *Let $\phi \in C(G)$. The following conditions are equivalent.*

- (i) ϕ is left G -finite;
- (ii) ϕ is right G -finite;
- (iii) there exist n and functions $a_i, b_i \in C(G)$, $1 \leq i \leq n$, such that

$$\phi(gh) = \sum_{i=1}^n a_i(g)b_i(h)$$

for all $g, h \in G$.

Proof. Let ϕ be a right G -finite function. Then ϕ is in a finite-dimensional subspace F invariant for R . The restriction of the representation R to F is continuous. Let a_1, a_2, \dots, a_n be a basis of F . Then, by 2.2.2, there exist $b_1, b_2, \dots, b_n \in C(G)$ such that $R(g)\phi = \sum_{i=1}^n b_i(g)a_i$. Therefore we have

$$\phi(hg) = \sum_{i=1}^n b_i(g)a_i(h) = \sum_{i=1}^n a_i(h)b_i(g)$$

for all $h, g \in G$. Therefore (iii) holds.

If (iii) holds,

$$R(g)\phi = \sum_{i=1}^n a_i(g)b_i$$

and ϕ is right G -finite.

Since the condition (iii) is symmetric, the equivalence of (i) and (iii) follows by applying the above argument to the opposite group of G . \square

Therefore, we can call ϕ just a G -finite function in $C(G)$. Let $R(G)$ be the subset of all G -finite functions in $C(G)$.

2.3.3. Proposition. *The set $R(G)$ is a subalgebra of $C(G)$.*

Proof. Clearly, a multiple of a G -finite function is a G -finite function.

Let ϕ and ψ be two G -finite functions. Then, by 2.3.2, there exists functions $a_i, b_i, c_i, d_i \in C(G)$ such that

$$\phi(gh) = \sum_{i=1}^n a_i(g)b_i(h) \text{ and } \psi(gh) = \sum_{i=1}^m c_i(g)d_i(h)$$

for all $g, h \in G$. This implies that

$$(\phi + \psi)(gh) = \sum_{i=1}^n a_i(g)b_i(h) + \sum_{i=1}^m c_i(g)d_i(h)$$

and

$$(\phi \cdot \psi)(gh) = \sum_{i=1}^n \sum_{j=1}^m a_i(g)c_j(g)b_i(h)d_j(h) = \sum_{i=1}^n \sum_{j=1}^m (a_i \cdot c_j)(g)(b_i \cdot d_j)(h)$$

for all $g, h \in G$. Hence, $\phi + \psi$ and $\phi \cdot \psi$ are G -finite. \square

Clearly, $R(G)$ is an invariant subspace for R and L .

The main result of this section is the following observation. Let V be a finite-dimensional complex linear space and π a continuous homomorphism of G into $\text{GL}(V)$, i.e., (π, V) is a *representation* of G . For $v \in V$ and $v^* \in V^*$ we call the continuous function $g \mapsto c_{v,v^*}(g) = \langle \pi(g)v, v^* \rangle$ a *matrix coefficient* of (π, V) .

2.3.4. Theorem. *Let $\phi \in C(G)$. Then the following statements are equivalent:*

- (i) ϕ is in $R(G)$;
- (ii) ϕ is a matrix coefficient of a finite-dimensional representation of G .

Proof. Let (π, V) be a finite-dimensional representation of G . Let $v \in V$ and $v^* \in V^*$. By scaling v^* if necessary, we can assume that v is a vector in a basis of V and v^* a vector in the dual basis of V^* . Then, $c_{v,v^*}(g)$ is a matrix coefficient of the matrix of $\pi(g)$ in the basis of V . The rule of matrix multiplication implies that (iii) from 2.3.2 holds for c_{v,v^*} . Hence ϕ is G -finite.

Assume that ϕ is G -finite. Then, by 2.3.2, we have $R(g)\phi = \sum_{i=1}^n a_i(g)b_i$ where $a_i, b_i \in C(G)$. We can also assume that b_i are linearly independent. Let V be the subspace of $R(G)$ spanned by b_1, b_2, \dots, b_n . Then V is a G -invariant subspace. Let $v = \phi$ and $v^* \in V^*$ such that $b_i(1) = \langle b_i, v^* \rangle$. Then

$$\langle R(g)v, v^* \rangle = \sum_{i=1}^n a_i(g)\langle b_i, v^* \rangle = \sum_{i=1}^n a_i(g)b_i(1) = \phi(g),$$

i.e., ϕ is a matrix coefficient of the restriction of R to V . \square

Therefore, we call $R(G)$ the *algebra of matrix coefficients* of G .

We also have the following stronger version of 2.3.2

2.3.5. Corollary. *Let $\phi \in R(G)$. Then there exist n and functions $a_i, b_i \in R(G)$, $1 \leq i \leq n$, such that*

$$\phi(gh) = \sum_{i=1}^n a_i(g)b_i(h)$$

for all $g, h \in G$.

Proof. Since ϕ is a matrix coefficient of a finite-dimensional representation by 2.3.4, the statement follows from the formula for the product of two matrices. \square

Moreover, $R(G)$ has the following properties. For a function $\phi \in C(G)$ we denote by $\bar{\phi}$ the function $g \mapsto \overline{f(g)}$ on G ; and by $\hat{\phi}$ the function $g \mapsto f(g^{-1})$.

2.3.6. Lemma. *Let $\phi \in R(G)$. Then*

- (i) the function $\bar{\phi}$ is in $R(G)$;
- (ii) the function $\hat{\phi}$ is in $R(G)$.

Proof. Obvious by 2.3.2. \square

3. SOME FACTS FROM FUNCTIONAL ANALYSIS

3.1. Compact operators. Let E be a Hilbert space and $T : E \rightarrow E$ a bounded linear operator.

We say that T is a *compact operator* if T is a bounded linear operator which maps the unit ball in E into a relatively compact set.

3.1.1. Lemma. *Compact operators form a two-sided ideal in the algebra of all bounded linear operators on E .*

Proof. Let S and T be compact operators. Let B be the unit ball in E . Then the images of B in E under T and S have compact closure. Hence, the image of $B \times B$ under $S \times T : E \times E \rightarrow E \times E$ has compact closure. Since the addition is a continuous map from $E \times E$ into E , the image of B under $S + T$ also has compact closure. Therefore, $S + T$ is a compact operator.

If S is a bounded linear operator and T a compact operator, the image of B under T has compact closure. Since S is continuous, the image of B under ST also has compact closure. Hence, ST is compact.

Analogously, the image of B under S is a bounded set since S is bounded. Therefore, the image of B under TS has compact closure and TS is also compact. \square

3.2. Compact selfadjoint operators. Let E be a Hilbert space. Let $T : E \rightarrow E$ be a nonzero compact selfadjoint operator.

3.2.1. Theorem. *Either $\|T\|$ or $-\|T\|$ is an eigenvalue of T .*

First we recall a simple fact.

3.2.2. Lemma. *Let u and v be two nonzero vectors in E such that $|(u|v)| = \|u\| \cdot \|v\|$. Then u and v are colinear.*

Proof. Let λv be the orthogonal projection of u to v . Then $u = \lambda v + w$ and w is perpendicular to v . This implies that $\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2$. On the other hand, we have $\|u\| \cdot \|v\| = |(u|v)| = |\lambda| \|v\|^2$, i.e., $|\lambda| = \frac{\|u\|}{\|v\|}$. Hence, it follows that

$$\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2 = \|u\|^2 + \|w\|^2,$$

i.e., $\|w\|^2 = 0$ and $w = 0$. \square

Now we can prove the theorem. By rescaling T , we can assume that $\|T\| = 1$.

Let B be the unit ball in E . By our assumption, we know that

$$1 = \|T\| = \sup_{v \in B} \|Tv\|.$$

Therefore, there exists a sequence of vectors $v_n \in B$ such that $\lim_{n \rightarrow \infty} \|Tv_n\| = 1$. Since T is compact, by going to a subsequence, we can also assume that $\lim_{n \rightarrow \infty} Tv_n = u$. This implies that

$$1 = \lim_{n \rightarrow \infty} \|Tv_n\| = \|u\|.$$

Moreover, we have $\lim_{n \rightarrow \infty} T^2 v_n = Tu$. Hence, we have

$$\begin{aligned} 1 = \|T\| \cdot \|u\| &\geq \|Tu\| = \lim_{n \rightarrow \infty} \|T^2 v_n\| \geq \limsup_{n \rightarrow \infty} (\|T^2 v_n\| \cdot \|v_n\|) \\ &\geq \limsup_{n \rightarrow \infty} (T^2 v_n | v_n) = \lim_{n \rightarrow \infty} (Tv_n | Tv_n) = \lim_{n \rightarrow \infty} \|Tv_n\|^2 = 1. \end{aligned}$$

It follows that

$$\|Tu\| = 1.$$

Moreover, we have

$$1 = \|Tu\|^2 = (Tu|Tu) = (T^2u|u) \leq \|T^2u\|\|u\| \leq \|T^2\|\|u\|^2 \leq \|T\|^2\|u\|^2 = 1.$$

This finally implies that

$$(T^2u|u) = \|T^2u\|\|u\|.$$

By 3.2.2, it follows that T^2u is proportional to u , i.e. $T^2u = \lambda u$. Moreover, we have

$$\lambda = \lambda(u|u) = (T^2u|u) = \|Tu\|^2 = 1.$$

It follows that $T^2u = u$.

Therefore, the linear subspace F of E spanned by u and Tu is T -invariant. Either $Tu = u$ or $v = \frac{1}{2}(u - Tu) \neq 0$. In the second case, we have $Tv = -v$.

This completes the proof of the existence of eigenvalues.

We need another fact.

3.2.3. Lemma. *Let T be a compact selfadjoint operator. Let λ be an eigenvalue different from 0. Then the eigenspace of λ is finite-dimensional.*

Proof. Assume that the corresponding eigenspace V is infinite-dimensional. Then there would exist an orthonormal sequence $(e_n, n \in \mathbb{N})$ in F . Clearly, then the sequence $(Te_n, n \in \mathbb{N})$ would consist of mutually orthogonal vectors of length $|\lambda|$, hence it could not have compact closure in V , contradicting the compactness of T . Therefore, V cannot be infinite-dimensional. \square

3.3. An example. Denote by μ the Haar measure on G . Let $L^2(G)$ be the Hilbert space of square-integrable complex valued functions on G with respect to the Haar measure μ . We denote its norm by $\|\cdot\|_2$. Clearly, we have

$$\|f\|_2^2 = \int_G |f(g)|^2 d\mu(g) \leq \|f\|^2$$

for any $f \in C(G)$. Hence the inclusion $C(G) \rightarrow L^2(G)$ is a continuous map.

3.3.1. Lemma. *The continuous linear map $i : C(G) \rightarrow L^2(G)$ is injective.*

Proof. Let $f \in C(G)$ be such that $i(f) = 0$. This implies that $\|f\|_2 = 0$. On the other hand, the function $g \mapsto |f(g)|$ is a nonnegative continuous function on G . Assume that M is the maximum of this function on G . If we would have $M > 0$, there would exist a nonempty open set $U \subset G$ such that $|f(g)| \geq \frac{M}{2}$ for $g \in U$. Therefore, we would have

$$\|f\|_2^2 = \int_G |f(g)|^2 d\mu(g) \geq \frac{M^2}{4} \mu(U) > 0,$$

by 1.3.16. Therefore, we must have $M = 0$. \square

Since the measure of G is 1, by Cauchy-Schwartz inequality, we have

$$\int_G |\phi(g)| d\mu(g) = \int_G 1 \cdot |\phi(g)| d\mu(g) \leq \|1\|_2 \cdot \|\phi\|_2 = \|\phi\|_2$$

for any $\phi \in L^2(\mu)$. Hence, $L_2(G) \subset L_1(G)$, where $L_1(G)$ is the Banach space of integrable functions on G .

Let f be a continuous function on G . For any $\phi \in L^2(G)$, we put

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)d\mu(h)$$

for $g \in G$.

By 1.1.1, f is uniformly continuous on G . This implies that for any $\epsilon > 0$ there exists a symmetric neighborhood U of 1 in G such that $g'g^{-1} \in U$ implies $|f(g) - f(g')| < \epsilon$. Therefore, for arbitrary $h \in G$, we see that for $(g'^{-1}h)(g^{-1}h)^{-1} = g'^{-1}g \in U$ and we have

$$|f(g^{-1}h) - f(g'^{-1}h)| < \epsilon.$$

This in turn implies that

$$\begin{aligned} |(R(f)\phi)(g) - (R(f)\phi)(g')| &= \left| \int_G f(h)\phi(gh)d\mu(h) - \int_G f(h)\phi(g'h)d\mu(h) \right| \\ &= \left| \int_G (f(g^{-1}h) - f(g'^{-1}h))\phi(h)d\mu(h) \right| = \int_G |f(g^{-1}h) - f(g'^{-1}h)| |\phi(h)| d\mu(h) \\ &< \epsilon \cdot \int_G |\phi(h)| d\mu(h) \leq \epsilon \cdot \|\phi\|_2 \end{aligned}$$

for any $g' \in Ug$ and ϕ in $L^2(G)$. This proves that functions $R(f)\phi$ are in $C(G)$ for any $\phi \in L^2(G)$.

Moreover, by the invariance of Haar measure, we have

$$\begin{aligned} |(R(f)\phi)(g)| &\leq \int_G |f(h)||\phi(gh)| d\mu(h) \leq \|f\| \int_G |\phi(gh)| d\mu(h) \\ &\leq \|f\| \int_G |\phi(h)| d\mu(h) \leq \|f\| \cdot \|\phi\|_2, \end{aligned}$$

it follows that

$$\|R(f)\phi\| \leq \|f\| \cdot \|\phi\|_2$$

for any $\phi \in L^2(G)$. Hence, $R(f)$ is a bounded linear operator from $L^2(G)$ into $C(G)$.

Hence the set $\mathcal{S} = \{R(f)\phi \mid \|\phi\|_2 \leq 1\}$ is bounded in $C(G)$.

Clearly, the composition of $R(f)$ with the natural inclusion $i : C(G) \rightarrow L^2(G)$ is a continuous linear map from $L^2(G)$ into itself which will denote by the same symbol. Therefore, the following diagram of continuous maps

$$\begin{array}{ccc} L^2(G) & \xrightarrow{R(f)} & L^2(G) \\ R(f) \downarrow & \nearrow i & \\ C(G) & & \end{array}$$

is commutative.

We already remarked that \mathcal{S} is a bounded set in $C(G)$. Hence, \mathcal{S} is a pointwise bounded family of continuous functions. In addition, by the above formula

$$|(R(f)\phi)(g) - (R(f)\phi)(g')| < \epsilon,$$

for all $g' \in Ug$ and ϕ in the unit ball in $L^2(G)$. Hence, the set \mathcal{S} is equicontinuous.

Hence we proved the following result.

3.3.2. Lemma. *The set $\mathcal{S} \subset C(G)$ is pointwise bounded and equicontinuous.*

By 1.2.1, the closure of the set \mathcal{S} in $C(G)$ is compact. Since $i : C(G) \rightarrow L^2(G)$ is continuous, \mathcal{S} has compact closure in $L^2(G)$. Therefore, we have the following result.

3.3.3. Lemma. *The linear operator $R(f) : L^2(G) \rightarrow L^2(G)$ is compact.*

Put $f^*(g) = \overline{f(g^{-1})}$, $g \in G$. Then $f^* \in C(G)$.

3.3.4. Lemma. *For any $f \in C(G)$ we have*

$$R(f)^* = R(f^*).$$

Proof. For $\phi, \psi \in L^2(G)$, we have, by 1.3.17,

$$\begin{aligned} (R(f)\phi \mid \psi) &= \int_G (R(f)\phi)(g) \overline{\psi(g)} d\mu(g) = \int_G \left(\int_G f(h)\phi(gh) d\mu(h) \right) \overline{\psi(g)} d\mu(g) \\ &= \int_G f(h) \left(\int_G \phi(gh) \overline{\psi(g)} d\mu(g) \right) d\mu(h) = \int_G f(h) \left(\int_G \phi(g) \overline{\psi(gh^{-1})} d\mu(g) \right) d\mu(h) \\ &= \int_G \phi(g) \left(\int_G \overline{f(h)} \psi(gh^{-1}) d\mu(h) \right) d\mu(g) \\ &= \int_G \phi(g) \left(\int_G f^*(h^{-1}) \psi(gh^{-1}) d\mu(h) \right) d\mu(g) \\ &= \int_G \phi(g) \left(\int_G f^*(h) \psi(gh) d\mu(h) \right) d\mu(g) = (\phi \mid R(f^*)\psi). \end{aligned}$$

□

3.3.5. Corollary. *The operator $R(f^*)R(f) = R(f)^*R(f)$ is a positive compact selfadjoint operator on $L^2(G)$.*

4. PETER-WEYL THEOREM

4.1. L^2 version. Let $\phi \in L^2(G)$. Let $g \in G$. We put $(R(g)\phi)(h) = \phi(hg)$ for any $h \in G$. Then we have

$$\|R(g)\phi\|_2^2 = \int_G |(R(g)\phi)(h)|^2 d\mu(h) = \int_G |\phi(hg)|^2 d\mu(h) = \int_G |\phi(h)|^2 d\mu(h) = \|\phi\|_2^2.$$

Therefore, $R(g)$ is a continuous linear operator on $L^2(G)$. Clearly it is in $\text{GL}(L^2(G))$. Moreover, $R(g)$ is unitary.

Clearly, for any $g \in G$, the following diagram

$$\begin{array}{ccc} C(G) & \xrightarrow{R(g)} & C(G) \\ i \downarrow & & \downarrow i \\ L^2(G) & \xrightarrow{R(g)} & L^2(G) \end{array}$$

is commutative.

Analogously, we define $(L(g)\phi)(h) = \phi(g^{-1}h)$ for $h \in G$. Then $L(g)$ is a unitary operator on $L^2(G)$ which extends from $C(G)$.

Clearly, $R(g)$ and $L(h)$ commute for any $g, h \in G$.

4.1.1. Lemma. *L and R are unitary representations of G on $L^2(G)$.*

Proof. It is enough to discuss R . The proof for L is analogous.

Let $g \in G$ and $\phi \in L^2(G)$. We have to show that $h \mapsto R(h)\phi$ is continuous at g . Let $\epsilon > 0$. Since $C(G)$ is dense in $L^2(G)$, there exists $\psi \in C(G)$ such that $\|\phi - \psi\|_2 < \frac{\epsilon}{3}$. Since R is a representation on $C(G)$, there exists a neighborhood U of g such that $h \in U$ implies $\|R(h)\psi - R(g)\psi\| < \frac{\epsilon}{3}$. This in turn implies that $\|R(h)\psi - R(g)\psi\|_2 < \frac{\epsilon}{3}$. Therefore we have

$$\begin{aligned} \|R(h)\phi - R(g)\phi\|_2 &\leq \|R(h)(\phi - \psi)\|_2 + \|R(h)\psi - R(g)\psi\|_2 + \|R(g)(\psi - \phi)\|_2 \\ &\leq 2\|\phi - \psi\|_2 + \|R(h)\psi - R(g)\psi\|_2 < \epsilon \end{aligned}$$

for any $h \in U$. \square

Let f be a continuous function on G . By 3.3.3, $R(f)$ is a compact operator on $L^2(G)$.

Let $\phi \in L^2(G)$. Then

$$\begin{aligned} (R(f)L(g)\phi)(h) &= \int_G f(k)(L(g)\phi)(hk) d\mu(k) \\ &= \int_G f(k)\phi(g^{-1}hk) d\mu(k) = (R(f)\phi)(g^{-1}h) = (L(g)R(f)\phi)(h) \end{aligned}$$

for all $g, h \in G$. Therefore, $R(f)$ commutes with $L(g)$ for any $g \in G$.

Let F be the eigenspace of $R(f^*)R(f)$ for eigenvalue $\lambda > 0$. Then F is finite-dimensional by 3.2.3.

- 4.1.2. Lemma.** (i) *Let $\phi \in F$. Then ϕ is a continuous function.*
(ii) *The vector subspace F of $C(G)$ is in $R(G)$.*

Proof. (i) The function ϕ is in the image of $R(f^*)$. Hence it is a continuous function.

(ii) By (i), $F \subset C(G)$. As we remarked above, the operator $R(f^*)R(f)$ commutes with the representation L . Therefore, the eigenspace F is invariant subspace for L . Let ϕ be a function in F . Since F is invariant for L , ϕ is G -finite. Hence, $\phi \in R(G)$. \square

4.1.3. Lemma. *The subspace $R(G)$ is invariant for $R(f)$.*

Proof. Let $\phi \in R(G)$. By 2.3.5 we have

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh) d\mu(h) = \sum_{i=1}^n a_i(g) \int_G f(h)b_i(h) d\mu(h)$$

for any $g \in G$, i.e., $R(f)\phi$ is a linear combination of a_i , $1 \leq i \leq n$. \square

Let $E = R(G)^\perp$ in $L^2(G)$. Then, by 4.1.3, $R(G)$ is invariant for selfadjoint operator $R(f^*)R(f)$. This in turn implies that E is also invariant for this operator. Therefore the restriction of this operator to E is a positive selfadjoint ocompact perator. Assume that its norm is greater than 0. Then, by 3.2.1, the norm is an eigenvalue of this operator, and there exists a nonzero eigenvector $\phi \in E$ for that eigenvalue. Clearly, ϕ is an eigenvector for $R(f^*)R(f)$ too. By 4.1.2, ϕ is also in $R(G)$. Hence, we have $\|\phi\|_2^2 = (\phi | \phi) = 0$, and $\phi = 0$ in $L^2(G)$. Hence, we have a contradiction.

Therefore, the operator $R(f^*)R(f)$ is 0 when restricted to E . Hence

$$0 = (R(f^*)R(f)\psi | \psi) = \|R(f)\psi\|^2$$

for any $\psi \in E$. It follows that $R(f)\psi = 0$. Since $R(f)\psi$ is a continuous function, we have

$$0 = (R(f)\psi)(1) = \int_G f(g)\psi(g)d\mu(g),$$

i.e., ψ is orthogonal to \bar{f} .

Since $f \in C(G)$ was arbitrary and $C(G)$ is dense in $L^2(G)$, it follows that $\psi = 0$. This implies that $E = \{0\}$.

This completes the proof of the following result.

4.1.4. Theorem (Peter-Weyl). *The algebra $R(G)$ is dense in $L^2(G)$.*

4.2. Continuous version. Let $g \in G$. Assume that $g \neq 1$. Then there exists an open neighborhood U of 1 such that U and Ug are disjoint. There exists positive function ϕ in $C(G)$ such that $\phi|_U = 0$ and $\phi|_{Ug} = 1$. This implies that

$$\begin{aligned} \|R(g)\phi - \phi\|^2 &= \int_G |\phi(hg) - \phi(h)|^2 d\mu(h) \\ &= \int_U |\phi(hg) - \phi(h)|^2 d\mu(h) + \int_{G-U} |\phi(hg) - \phi(h)|^2 d\mu(h) \geq \mu(U). \end{aligned}$$

Therefore $R(g) \neq I$. Since by 4.1.4, $R(G)$ is dense in $L^2(G)$, $R(g)|_{R(G)}$ is not the identity operator.

This implies the following result.

4.2.1. Lemma. *Let $g, g' \in G$ and $g \neq g'$. Then there exists a function $\phi \in R(G)$ such that $\phi(g) \neq \phi(g')$.*

Proof. Let $h = g^{-1}g' \neq 1$. Then there exists $\psi \in R(G)$ such that $R(h)\psi \neq \psi$. Hence, we have $R(g)\psi \neq R(g')\psi$. It follows that $\psi(hg) \neq \psi(hg')$ for some $h \in G$. Therefore, the function $\phi = L(h^{-1})\psi$ has the required property. \square

In other words, $R(G)$ separates points in G . By Stone-Weierstrass theorem, we have the following result which is a continuous version of Peter-Weyl theorem.

4.2.2. Theorem (Peter-Weyl). *The algebra $R(G)$ is dense in $C(G)$.*

Another consequence of 4.2.1 is the following result.

4.2.3. Lemma. *Let U be an open neighborhood of 1 in G . Then there exists a finite-dimensional representation (π, V) of G such that $\ker \pi \subset U$.*

Proof. The complement $G - U$ of U is a compact set. Since $R(G)$ separates the points of G , for any $g \in G - U$ there exists a function $\phi_g \in R(G)$ and an open neighborhood U_g of g such that $|\phi_g(h) - \phi_g(1)| > \epsilon$ for $h \in U_g$. Since $G - U$ is compact, there exists a finite set g_1, g_2, \dots, g_m in $G - U$ such that $U_{g_1}, U_{g_2}, \dots, U_{g_m}$ form an open cover of $G - U$ and $|\phi_{g_i}(h) - \phi_{g_i}(1)| > \epsilon$ for $h \in U_{g_i}$. Let π_i be a finite-dimensional representation of G with matrix coefficient ϕ_{g_i} . Then $\pi_i(h) \neq I$ for $h \in U_{g_i}$, $1 \leq i \leq m$. Let π be the direct sum of π_i . Then $\pi(h) \neq I$ for $h \in G - U$, i.e., $\ker \pi \subset U$. \square

4.3. Matrix groups. Let G be a topological group. We say that G has no small subgroups if there exists a neighborhood U of $1 \in G$ such that any subgroup of G contained in U is trivial.

4.3.1. Lemma. *Let V be a finite-dimensional complex vector space. Then the group $\mathrm{GL}(V)$ has no small subgroups.*

Proof. Let $\mathcal{L}(V)$ be the space of all linear endomorphisms of V . Then $\exp : \mathcal{L}(V) \rightarrow \mathrm{GL}(V)$ given by

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

defines a holomorphic map. Its differential at 0 is the identity map I on $\mathcal{L}(V)$. Hence, by the inverse function theorem, it is a local diffeomorphism.

Let U be an open neighborhood of 1 in $\mathrm{GL}(V)$ and V the open ball around 0 in $\mathcal{L}(V)$ of radius ϵ (with respect to the linear operator norm) such that $\exp : V \rightarrow U$ is a diffeomorphism. Let V' be the open ball of radius $\frac{\epsilon}{2}$ around 0 in $\mathcal{L}(V)$. Then $U' = \exp(V')$ is an open neighborhood of 1 in $\mathrm{GL}(V)$. Let H be a subgroup of $\mathrm{GL}(V)$ contained in U' . Let $S \in H$. Then $S = \exp(T)$ for some $T \in V'$. Hence, we have $S^2 = \exp(T)^2 = \exp(2T) \in H$. Moreover, $S^2 \in H$ and $S^2 = \exp(T')$ for some $T' \in V'$. It follows that $\exp(T') = \exp(2T)$ for $2T, T' \in V$. Since \exp is injective on V , we must have $2T = T'$. Hence, $T \in \frac{1}{2}V'$. It follows that $H \subset \exp(\frac{1}{2}V')$. By induction we get that $H \subset \exp(\frac{1}{2^n}V')$ for any $n \in \mathbb{N}$. This implies that $H = \{1\}$. \square

A compact subgroup of $\mathrm{GL}(V)$ we call a compact *matrix* group.

4.3.2. Theorem. *Let G be a compact group. Then the following conditions are equivalent:*

- (i) G has no small subgroups;
- (ii) G is isomorphic to a compact matrix group.

Proof. (i) \Rightarrow (ii) Let U be an open neighborhood of $1 \in G$ such that it contains no nontrivial subgroups of G . By 4.2.3, there exists a finite-dimensional representation (π, V) of G such that $\ker \pi \subset U$. This clearly implies that $\ker \pi = \{1\}$, and $\pi : G \rightarrow \mathrm{GL}(V)$ is an injective homomorphism. Since G is compact, π is homeomorphism of G onto $\pi(G)$. Therefore, G is isomorphic to the compact subgroup $\pi(G)$ of $\mathrm{GL}(V)$.

(ii) \Rightarrow (i) Assume that G is a compact subgroup of $\mathrm{GL}(V)$. By 4.3.1, there exists an open neighborhood U of 1 in $\mathrm{GL}(V)$ such that it contains no nontrivial subgroups. This implies that $G \cap U$ contains no nontrivial subgroups of G . \square

4.3.3. Remark. For a compact matrix group G , since matrix coefficients of the natural representation separate points in G , 4.2.1 obviously holds. Therefore, in this situation, Stone-Weierstrass theorem immediately implies the second version of Peter-Weyl theorem, which in turn implies the first one.

4.3.4. Remark. By Cartan's theorem [?], any compact matrix group is a Lie group. On the other hand, by [?] any Lie group has no small subgroups. Hence, compact Lie groups have no small subgroups and therefore they are compact matrix groups.

4.3.5. Remark. Let $T = \mathbb{R}/\mathbb{Z}$. Then T is a compact abelian group. Let G be the product of infinite number of copies of T . Then G is a compact abelian group. By the definition of product topology, any neighborhood of 1 contains a nontrivial subgroup.

Let G be an arbitrary compact group. Let (π, V) be a finite-dimensional representation. Put $N = \ker \pi$. Then N is a compact normal subgroup of G and G/N equipped with the quotient topology is a compact group. Clearly, G/N is a compact matrix group.

Let \mathcal{S} be the family of all compact normal subgroups N of G such that G/N is a compact matrix group. Clearly, N, N' in \mathcal{S} implies $N \cap N' \in \mathcal{S}$. Therefore, \mathcal{S} ordered by inclusion is a directed set. One can show that G is a projective limit of the system $G/N, N \in \mathcal{S}$. Therefore, any compact group is a projective limit of compact matrix groups. By the above remark, this implies that any compact group is a projective limit of compact Lie groups.