Math 6220-1 Midterm 2, April 16, 2012

1. Using the residue theorem evaluate the integral

$$\int_0^\infty \frac{\log x}{x^2 + a^2}\, dx$$

where $a \neq 0$.

2. Using the residue theorem evaluate the integral

$$\int_0^\infty \frac{x^p}{1 + x^2}\, dx$$

where $-1 < p < 1$.

3. Using Rouche's theorem find the number of zeros of the polynomial

$$z^9 - 2z^6 + z^2 - 8z - 2$$

in the unit disk around 0.

4. Let S be a Riemann surface and $f : S \longrightarrow \mathbb{C}$ a holomorphic function. Assume that a is an isolated zero of f. Let (U, ϕ) and (U', ϕ') be two charts centered at a. Then the holomorphic functions $f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{C}$ and $f \circ (\phi')^{-1} : \phi'(U') \longrightarrow \mathbb{C}$ have isolated zeros at 0. Show that their orders are equal.

We say that the common order is the *order* of zero a of f.

5. Let S be a Riemann surface, a a point in S and $f : S - \{a\} \longrightarrow \mathbb{C}$ a holomorphic function. Let (U, ϕ) and (U', ϕ') be two charts centered at a. Assume that the holomorphic functions $f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{C}$ and $f \circ (\phi')^{-1} : \phi'(U') \longrightarrow \mathbb{C}$ have poles at 0. Show that their orders are equal.

We say that the common order is the *order* of zero a of f.

6. Let S be a Riemann surface, a a point in S and $f : S - \{a\} \longrightarrow \mathbb{C}$ a holomorphic function. Assume that a is a pole of f. If we identify $\mathbb{C}$ with the complement of ∞ in the Riemann sphere $\mathbb{P}^1$, we can view f as a holomorphic map into $\mathbb{P}^1$, and extend it to a map $F : S \longrightarrow \mathbb{P}^1$ such that $F(a) = \infty$. Show that F is a holomorphic map.

7. If we identify the complex plane $\mathbb{C}$ with the complement of ∞ in the Riemann sphere $\mathbb{P}^1$, we can view polynomials as holomorphic functions on $\mathbb{P}^1 - \{\infty\}$. Show:

(a) These polynomials have a pole at ∞.

(b) In this way, we see that we can view rational functions on $\mathbb{C}$ as meromorphic functions on $\mathbb{P}^1$. Show that every meromorphic function on $\mathbb{P}^1$ is rational.