1. Determine if there exists an entire function such that
   \[ f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n} \]
   for all \( n \in \mathbb{N} \).

2. Let \( f \) be an entire function. Suppose that in its power series expansions
   \[ f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \]
   for any \( a \in \mathbb{C} \), at least one coefficient is equal to zero (where \( n \) can depend on \( a \)). Show that \( f \) is a polynomial.

3. Suppose that \( f \) and \( g \) are entire functions such that \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{C} \). What conclusions can you draw?

4. Let \( f(z) = 1 - \cos z \).
   
   (i) Find all zeros of this function;
   
   (ii) find the multiplicities of these zeros.

In the next three problems we develop the basic facts about Laurent series. A Laurent series
   \[ \sum_{n=-\infty}^{\infty} c_n(z-a)^n \]
around \( a \in \mathbb{C} \) is the sum of the series
   \[ \sum_{n=0}^{\infty} \frac{c_{-n}}{(z-a)^n} \text{ and } \sum_{n=0}^{\infty} c_n(z-a)^n. \]

The first series is called the principal part and the second the regular part of the Laurent series.

5. Let
   \[ \sum_{n=-\infty}^{\infty} c_n(z-a)^n \]
be a Laurent series. Let
   \[ r = \limsup_{n \to \infty} |c_{-n}|^{\frac{1}{n}} \quad \text{and} \quad R = \frac{1}{\limsup_{n \to \infty} |c_n|^{\frac{1}{n}}} \]
Then the Laurent series converges absolutely in the open annulus \( \{ z \in \mathbb{C} \mid r < |z - a| < R \} \) and diverges for \( |z - a| < r \) and \( |z - a| > R \). If the above annulus is nonempty, the function
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n
\]
is holomorphic in the annulus.

6. Let \( f \) be a holomorphic function in the open annulus \( \{ z \in \mathbb{C} \mid r < |z - a| < R \} \). Let \( \gamma \) be a positively oriented circle centered at \( a \) of radius \( \rho \) such that \( r < \rho < R \). Put
\[
c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{n+1}} \, dz
\]
for any \( n \in \mathbb{Z} \). Show that
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n
\]
in the annulus (This series is called the Laurent series of \( f \)).

7. Let \( \Omega \) be a domain and \( a \in \Omega \). Let \( f \) be a function holomorphic in \( \Omega - \{a\} \). Let \( D(a, R) \) be an open disk in \( \Omega \). Then \( f \) can be represented by its Laurent series
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n
\]
on the punctured disk \( D'(a, R) \). Show:
(a) \( a \) is a removable singularity if and only if \( c_n = 0 \) for all \( n < 0 \);
(b) \( a \) is a pole of order \( m \) if and only if \( c_{-m} \neq 0 \) and \( c_n = 0 \) for \( n < -m \);
(c) \( a \) is an essential singularity if and only if infinitely many \( c_n \) are different from 0 for \( n < 0 \).

8. Let
\[
f(z) = \sin \left( \frac{z}{z + 1} \right).
\]
(i) Determine all isolated singularities of \( f \) and their type;
(ii) find the Laurent expansions of \( f \) at these singularities;
(iii) find the residues of \( f \) at these singularities.

9. Evaluate the integral
\[
\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 2x + 10} \, dx
\]
using the residue theorem.

10. Evaluate the integral

\[ \int_{0}^{2\pi} \frac{\cos^2 3\phi}{1 - 2a \cos \phi + a^2} d\phi \]

where \( a \) is a complex number such that \( |a| < 1 \), using the residue theorem.