

1. Let f be an entire function. Suppose that in its power series expansions

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

for any $a \in \mathbb{C}$, at least one coefficient c_n is equal to zero (where n can depend on a). Show that f is a polynomial.

2. Let S be a domain and f a holomorphic function on S . Assume that $a \in S$ is a zero of order m of f and $f(z) \neq 0$ for $z \neq a$. Show that $g = \frac{1}{f}$ has a pole of order m at a .
3. Suppose that f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. What conclusion can you draw?

In the next few problems develop the basic facts about *Laurent series*. A Laurent series

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

around $a \in \mathbb{C}$ is the sum of the series

$$\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} \text{ and } \sum_{n=0}^{\infty} c_n(z-a)^n.$$

The first series is called *the principal part* and the second *the regular part* of the Laurent series.

4. Let

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

be a Laurent series. Let

$$r = \liminf_{n \rightarrow \infty} |c_{-n}|^{\frac{1}{n}} \text{ and } R = \frac{1}{\liminf_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}.$$

Then the Laurent series converges absolutely in the open annulus $\{z \in \mathbb{C} \mid r < |z - a| < R\}$ and diverges for $|z - a| < r$ and $|z - a| > R$. If the above annulus is nonempty, the function

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - a)^n$$

is holomorphic in the annulus.

5. Let f be a holomorphic function in the open annulus $\{z \in \mathbb{C} \mid r < |z - a| < R\}$. Let γ be a positively oriented circle centered at a of radius ρ such that $r < \rho < R$. Put

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{n+1}} dz$$

for any $n \in \mathbb{Z}$. Show that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - a)^n$$

in the annulus. (This series is called the *Laurent series* of f .)

6. Let S be a domain and $a \in S$. Let f be a function holomorphic in $S - \{a\}$. Let $D(a, R)$ be an open disk in S . Then f can be represented by its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - a)^n$$

for $z \in D'(a, R)$. Show:

- (a) a is a removable singularity if and only if $c_n = 0$ for all $n < 0$;
 - (b) a is a pole of order m if and only if $c_{-m} \neq 0$ and $c_n = 0$ for $n < -m$;
 - (c) a is an essential singularity if and only if infinitely many c_n are different from 0 for $n < 0$.
7. Find the Laurent series of $f(z) = e^{\frac{1}{z}}$ at $a = 0$. What kind of singularity is $z = 0$ for f ?

8. Let a be an isolated singularity of a holomorphic function f . Let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

be its Laurent series. Show that $\text{Res}(f, a) = c_{-1}$.

9. Using the residue theorem evaluate the integral

$$\int_0^{\infty} \frac{x^2 + 1}{x^4 + 1} dx.$$

10. Using the residue theorem evaluate the integral

$$\int_0^{\infty} \frac{\sin(ax)}{x(x^2 + b^2)^2} dx$$

where a and b are nonzero real numbers. (Hint: Use Jordan lemma!)