Math 3220-3 Take Home Midterm 3, April 10, 2021

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Name:

Problem 1. If F is a differentiable real function defined in a convex open set $U \subset \mathbb{R}^n$, such that $\partial_1 F(x) = 0$ for every $x \in U$, prove that F depends only on $x_2, \ldots, x_n$.

Problem 2. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map given by $F = (F_1, F_2)$ where

$$F_1(x, y) = e^x \cos(y) \quad \text{and} \quad F_2(x, y) = e^x \sin(y)$$

for any $(x, y) \in \mathbb{R}^2$. Then:

(i) Find the image of F.

(ii) Calculate the differential $F'(x, y)$ and show that it is invertible at any point in $\mathbb{R}^2$.

Thus, by the inverse function theorem, F is locally invertible, i.e., for any $(x, y) \in \mathbb{R}^2$ there are open neighborhoods U of (x, y) and V of $F(x, y)$ such that $F : U \longrightarrow V$ is a bijection.

Show that F is not a bijection globally, i.e, F is not a bijection of $\mathbb{R}^2$ onto the image of F.

Problem 3. Let f be a function on $\mathbb{R}$ defined by

$$f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$$

for $x \neq 0$ and $f(0) = 0$. Show that

(i) f is continuous on $\mathbb{R}$;

(ii) f is differentiable on $\mathbb{R}$;

(iii) the derivative f' is not continuous at 0;

(iv) $f'(0) = 1$;

(v) for any $\epsilon > 0$, the restriction of f to $(-\epsilon, \epsilon)$ is not injective.

This shows that, even for $n = 1$, the conclusions of inverse function theorem do not hold if f' is not continuous.

Hint: To prove (v), first show that a continuous function f cannot be injective in neighborhoods of local maxima and minima.

These must be critical points of f, i.e. zeros of f'.

Then show that for every $\epsilon > 0$ the interval $(-\epsilon, \epsilon)$ contains infinitely many critical points of f.

A critical point x of f is a maximum or minimum if $f''(x) \neq 0$.

Therefore, it is enough to show that there is an $\epsilon > 0$ such that there are no $x \in (-\epsilon, \epsilon)$ such that $f'(x) = 0$ and $f''(x) = 0$.

To prove this observe that the derivatives f' and f'' are linear functions in $A = \sin\left(\frac{1}{x}\right)$ and $B = \cos\left(\frac{1}{x}\right)$ with coefficients which are rational functions in x. Therefore, the equations $f'(x) = 0$ and $f''(x) = 0$ are a linear system of two equations for A and B with rational function coefficients.

Explicitly solve this system for A and B. Then calculate $A^2 + B^2$. From the result you should see that for small x this expression cannot be 1, contradicting the choice of A and B. Therefore, we have a contradiction. Hence, for small x, f' and f'' cannot simultaneously vanish at x.

Problem 4. Define

$$F(x, y) = (e^x \cos y - 1, e^x \sin y)$$

for all $(x, y) \in \mathbb{R}^2$. Prove that $F = G_2 \circ G_1$, where

$$G_1(x, y) = (e^x \cos y - 1, y) \text{ and } G_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$.