**Change of variables**

Let $U, V$ open sets in $\mathbb{R}^n$

$T : U \to V$ continuously differentiable bijection

Then $T'(x)$ is a continuous function on $U$

\[ J_T(x) = \det T'(x) \]

is a continuous function on $U$.

Assume that $J_T(x) \neq 0$ for all $x \in U$. 
them $T'(x)$ is invertible for any $x \in U$. Therefore, by inverse function theorem, the inverse function $T^{-1}: V \to U$ is continuously differentiable. Moreover,

$$J_{T^{-1}}(T(x)) = \frac{\det(T'(x))}{\det(T'(x))} = \frac{1}{\det(T'(x))} = J_T(x).$$

Since $T^{-1}$ is continuous, it maps compact sets into compact sets.
Let \( f \in C_0(\mathbb{R}^n) \), \( \text{supp} \ f \subset V^3 \).

Then \( f \circ T \) is a continuous function on \( U \).

If \( K = \text{supp} \ f \), \( f(T(x)) \neq 0 \) implies \( T(x) \in K \), i.e. \( x \in T^{-1}(K) \). As we remarked, \( T^{-1}(K) \) is compact.

Therefore, \( \text{supp}(f \circ T) \) is a closed subset of \( T^{-1}(K) \), i.e. it is compact.

We want to prove the following formula:
\[ \int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} f(T(y)) |J_T(y)| \, dy \]

This is the change of variables formula.

Before proving the formula we discuss the case of \( m = 1 \).

1 and \( J \) are open intervals, \( T \) is continuously differentiable.
Since $T'(x) \neq 0$ for all $x \in I$ we have two cases.

Either (a) $T'(x) > 0$ for all $x \in I$ or (b) $T'(x) < 0$ for all $x \in I$.

In the case (a) $T$ is strictly increasing. Therefore $T(a) = c$ and $T(b) = d$.

\[
\begin{align*}
\int_{a}^{b} f(T(x)) T'(x) \, dx &= \int_{a}^{d} f(x) \, dx \\
&= \int_{c}^{d} f(x) \, dx
\end{align*}
\]

In the second case, $T$ is strictly decreasing. Therefore $T(a) = d$ and $T(b) = c$. 
\[
\int_a^b f(T(x)) T'(x) \, dx = \int_c^d f(x) \, dx = -\int_c^d f(x) \, dx.
\]

Hence,
\[
\int_c^d f(x) \, dx = \int_a^b f(T(x)) |T'(x)| \, dx.
\]

Hence in both cases
\[
\int_R^R f(x) \, dx = \int_R^R f(T(x)) |T'(x)| \, dx,
\]

and this is a special case of change of variables formula.
First reduction

Assume that $T$ and $S$ are continuously differentiable bijections of $U$ onto $V$, and $V$ onto $W$ respectively.

Also assume that

\[ JT(x) \neq 0 \text{ for all } x \in U \]
\[ J_S(y) \neq 0 \text{ for all } y \in V. \]

Then $P = S \circ T$ is a continuous bijection of $U$ onto $W$. Moreover,
by chain rule
\[ P'(x) = S'(T(x)) \cdot T'(x) \]
for any \( x \in U \). It follows that
\[
J_p(x) = \det P'(x) = \\
= \det (S'(T(x))) \cdot T'(x) = \\
= \det (S'(T(x))) \cdot \det T'(x) = \\
= J_S(T(x)) \cdot J_T(x)
\]
for \( x \in U \).

Let \( f \) be a continuous function with compact support in \( W \). Then \( f \circ S \) has compact support in \( V \), and \( f \circ P \) has compact support.
Support in $U$. Therefore

\[ \int_{\mathbb{R}^n} f(P(x)) |J_p(x)| \, dx = \]

\[ = \int_{\mathbb{R}^n} f(S(T(x)) |J_p(T(x))| \, dx = \]

\[ = \int_{\mathbb{R}} f(S(y)) |J_p(y)| \, dy = \]

(Using the change of variables formula for $S$)

\[ = \int_{\mathbb{R}} f(z) \, dz \]

(Using the change of variables formula for $T$)
Therefore if \( P = S \circ T \) and the formula holds for \( T \) and \( S \), it also holds for their composition \( P = S \circ T \).

Second reduction

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{T} & \mathcal{V} \\
\mathcal{C} & \xrightarrow{f} & \mathcal{V}
\end{array}
\]

Assume that for any \( a \in \mathcal{V} \) there exists an open neighborhood \( V_a \) of \( a \), \( V_a \subset \mathcal{V} \) such that the formula holds for functions \( f \in C_0(\mathbb{R}^m) \) such that
Let $f \in C_0(\mathbb{R}^m)$ with $K = \text{supp } f \subset V$. Then $(V_\alpha; \alpha \in K)$ is an open cover of $K$. Let

$(\psi_1, \ldots, \psi_m)$ be a partition of unity subordinated to the cover $(V_\alpha; \alpha \in K)$.

Then

$$\sum_{i=1}^{m} \psi_i(y) = 1 \text{ for } y \in K.$$
Put $f_i = f \cdot q_i$.

Then $\text{supp } f_i \subset V_{a_i}$ for some $a_i \in \mathbb{K}$.

By the assumption, we have

$$
\int_{\mathbb{R}^n} f_i(\tau(x)) \left| J_\tau(x) \right| \, dx =
$$

$$
= \int_{\mathbb{R}^n} f_i(x) \, dx
$$

for $1 \leq i \leq m$. Hence

$$
\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} \sum_{i=1}^{m} f_i(x) \, dx =
$$

$$
= \sum_{i=1}^{m} \int_{\mathbb{R}^n} f_i(x) \, dx =
$$

$$
= \sum_{i=1}^{m} \int_{\mathbb{R}^n} f_i(\tau(x)) \left| J_\tau(x) \right| \, dx
$$
\[
\int_{\mathbb{R}^n} \sum_{i=1}^{m} f_i(T(x)) |J_{f_i}(x)| \, dx = 
\int_{\mathbb{R}^n} f(T(x)) |J_{f_i}(x)| \, dx 
\]

So, the formula holds for \( f \).

(This is a typical example of reducing a global statement to local using partition of unity.)